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Elements of Numerical Relativity and Relativistic Hydrodynamics

From Einstein's Equations to Astrophysical Simulations

Second Edition

Springer

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To Montse, my dear wife and friend.

Para Tania, mi amor.

Preface to the Second Edition

The first edition of this book was issued by 2005, with the objective of providing basic tools for beginning graduate students interested in numerical relativity research. The size of the numerical relativity community has experienced a significant increase since then, due to the scientific breakthroughs in binary black hole simulations which started precisely by autumn 2005 with the famous Pretorius work. Perhaps this has contributed to exhaust the first printed edition in a couple of years.

This second edition provides the opportunity to include important new developments that have arisen since 2005, which we detail below. It will also be the opportunity to respond to the continuing shift in the community scientific objectives, by incorporating a new chapter on relativistic hydrodynamics and magnetohydrodynamics. We have tried to keep the focus on basic tools and formalisms, with most numerical applications being able to run in a single PC. But proper reference is also made to more advanced developments, requiring much larger computational resources.

Here is the list of the main changes and additions:

- In the first edition, there was no description of any harmonic formalism whatsoever. It was justified because this approach was not a mainstream one in 3D numerical relativity applications at that time. But things changed suddenly when Pretorius result happened to be precisely in a generalized harmonic formulation. The leading groups immediately tried to follow the same way, with diverse results. Today, both BSSN and generalized harmonic formulations are first-rank options in current binary black hole simulations. This material has been added as a new section at the end of the first chapter, dealing with the structure of the field equations. The important point of the mutual relationship between the harmonic and Z4 evolution formalisms is discussed in the third chapter, in a new section dealing with covariant formulations. Moreover, it has been possible recently to match numerical results with analytical approximations (in harmonic coordinates) for black hole simulations. This is why we include also at the end of the first chapter a concise account of approximate solution methods, which were just mentioned in the first edition.

- The Z4 formalism was used for deriving by mid-2005 (when the first edition was yet in print) some convenient damping terms for the energy–momentum constraints, together with their translation into the generalized harmonic framework. These new damping terms, which were actually incorporated in Pretorius work, are properly introduced in Chap. 3. Also, the ordering constraints arising in the passage from a second-order to a first-order (in space) system deserve a enhanced discussion in Chap. 4, in particular, the ordering constraints related with the shift derivatives, which were overlooked in the first edition. They have later shown their importance in the passage from the second-order generalized harmonic formalisms to their first-order version, with the inclusion of specific damping terms.
- We have added a new chapter dealing with non-vacuum spacetimes. It starts with the scalar field case, which has been considered as a candidate for modeling dark matter. Then we follow with sections on electromagnetic fields and on relativistic hydrodynamics. This sets the basis for the magnetohydrodynamics section, where we consider the general case, even beyond the ideal MHD one. This is a deliberate choice, as we feel that new important developments will come precisely in this area, contributing to the full explanation of many puzzling astrophysical observations.
- Concerning numerical tools, finite-volume methods should be still considered, with a view on hydrodynamical simulations. In the first edition, however, an upwind-biased variant was proposed, which required using the full eigenvector decomposition. This is not the mainstream practice today, specially in MHD applications, where the expressions for the eigenvectors get really complicated. The community is rather moving toward centered flux formulae, much more cost-efficient. In the case of the spacetime evolution, where just smooth solutions are expected, some finite-differences versions of these methods can be used with a minimal computational cost, keeping most of the robustness of the original finite-volume algorithms. Numerical methods are now included in a new specific chapter. These new tools allow for long-term black hole simulations even in normal coordinates, as described in Chap. 6.

Palma de Mallorca,
February 2009

Carles Bona
Carlos Palenzuela Luque
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Preface to the First Edition

We got involved with numerical relativity under very different circumstances. For one of us (CB) it dates from about 1987, when the current Laser-Interferometer Gravitational Wave Observatories were just promising proposals. It was during a visit to Paris, at the Institut Henri Poincaré, where some colleagues were pushing the VIRGO proposal with such a contagious enthusiasm that I actually decided to reorient my career. The goal was to be ready, armed with a reliable numerical code, when the first detection data would arrive.

Allowing for my experience with the 3+1 formalism at that time, I started working on singularity-avoidant gauge conditions. Soon, I became interested in hyperbolic evolution formalisms. When trying to get some practical applications, I turned upon numerical algorithms (a really big step for a theoretically oriented guy) and black hole initial data. More recently, I got interested in boundary conditions and, closing the circle, again in gauge conditions. The problem is that a reliable code needs all these ingredients working fine at the same time. It is like an orchestra, where strings, woodwinds, brass, and percussion must play together in a harmonic way: a violin virtuoso, no matter how good, cannot play Vivaldi's Four Seasons by himself.

During that time, I have got many Ph.D. students. The most recent one is the other of us (CP). All of them started with some specific topic, but they needed a basic knowledge of all the remaining ones: you cannot work on the saxo part unless you know what the bass is supposed to play at the same time.

This is where this book can be of a great help. Imagine a beginning graduate student armed just with a home PC. Imagine that the objective is to build a working numerical code for simple black hole applications. The book should provide him or her with a basic insight on the most relevant aspects of numerical relativity in the first place. But this is not enough, the book should also provide reliable and compatible choices for every component: evolution system, gauge, initial and boundary conditions, even for the numerical algorithms.

This pragmatical orientation may cause this book to be seen as biased. But the idea was not to get a compendium of the excellent work that has been made on numerical relativity during these years. The idea is rather to present a well-founded and convenient way for a beginner to get into the field. He or she will quickly discover everything else.

The structure of the book reflects the peculiarities of numerical relativity research:

- It is strongly rooted in theory. Einstein's relativity is a general covariant theory. This means that we are building at the same time the solution and the coordinate system, a unique fact among physical theories. This point is stressed in the first chapter, which could be omitted by more experienced readers.
- It turns the theory upside down. General covariance implies that no specific coordinate is more special that the others, at least not a priori. But this is at odds with the way humans and computers usually model things: as functions (of space) that evolve in time. The second chapter is devoted to the evolution (or 3+1) formalism, which reconciles general relativity with our everyday perception of reality, in which time plays such a distinct role.
- It is a fertile domain, even from the theoretical point of view. The structure of Einstein's equations allows many ways of building well-posed evolution formalisms. Chapter 3 is devoted to those which are of first order in time but second order in space. Chapter 4 is devoted instead to those which are of first order both in time and in space. In both cases, suitable numerical algorithms are provided, although the most advanced ones apply mainly to the fully first-order case.
- It is challenging. The last sections of Chaps. 5 and 6 contain front-edge developments on constraint-preserving boundary conditions and gauge pathologies, respectively.[1] These are very active research topics, where new developments will soon improve the ones presented here. The prudent reader is encouraged to look for updates of these front-edge parts in the current scientific literature.

A final word. Numerical relativity is not a matter of brute force. Just a PC, not a supercomputer, is required to perform the tests and applications proposed here. Numerical relativity is instead a matter of insight. Let the wisdom be with you.

Palma de Mallorca, *Carles Bona*
January 2005 *Carlos Palenzuela Luque*

[1] Note to the second edition. The chapter numbers here correspond to the first edition. In this second edition, these tentative developments have been either removed or replaced by other material. This fact confirms the prediction we made in this first Preface.

Contents

1 The 4D Spacetime ... 1
1.1 Spacetime geometry ... 1
1.1.1 The metric ... 1
1.1.2 General covariance ... 2
1.1.3 Covariant derivatives ... 3
1.1.4 Curvature ... 5
1.1.5 Symmetries of the curvature tensor ... 6
1.2 General covariant field equations ... 7
1.2.1 The stress–energy tensor ... 7
1.2.2 Einstein's field equations ... 8
1.2.3 Structure of the field equations ... 10
1.3 Einstein's equations solutions ... 13
1.3.1 Symmetries: Lie derivatives ... 13
1.3.2 Exact solutions ... 15
1.3.3 Analytical and numerical approximations ... 16
1.4 Harmonic formalism ... 18
1.4.1 The relaxed system ... 19
1.4.2 Analytical and numerical applications ... 20
1.4.3 Harmonic coordinates ... 22
References ... 24

2 The Evolution Formalism ... 25
2.1 Space plus time decomposition ... 25
2.1.1 A prelude: Maxwell equations ... 26
2.1.2 Spacetime synchronization ... 27
2.1.3 The Eulerian observers ... 30
2.2 Einstein's equations decomposition ... 32
2.2.1 The 3+1 form of the field equations ... 32
2.2.2 3+1 Covariance ... 33
2.2.3 Generic space coordinates ... 35
2.3 The evolution system ... 38

2.3.1 Evolution and constraints ... 38
2.3.2 Constraints conservation ... 39
2.3.3 Evolution strategies ... 40
2.4 Gravitational waves degrees of freedom ... 42
2.4.1 Linearized field equations ... 42
2.4.2 Plane-wave analysis ... 43
2.4.3 Gravitational waves and gauge effects ... 46
References ... 47

3 Free Evolution ... 49
3.1 The free evolution framework ... 49
3.1.1 The ADM system ... 49
3.1.2 Extended solution space ... 50
3.1.3 Plane-wave analysis ... 52
3.2 Robust stability test-bed ... 55
3.2.1 Finite differences ... 55
3.2.2 Numerical results ... 58
3.3 Pseudo-hyperbolic systems ... 60
3.3.1 Extra dynamical fields ... 60
3.3.2 The BSSN system ... 63
3.3.3 Plane-wave analysis ... 65
3.4 Covariant formulations ... 67
3.4.1 The Z4 formalism ... 67
3.4.2 The generalized harmonic formalism ... 69
3.4.3 Constraint-violation control ... 70
3.5 The Z4 evolution system ... 71
3.5.1 $3+1$ Decomposition ... 71
3.5.2 Plane-wave analysis ... 73
3.5.3 Symmetry breaking ... 75
References ... 77

4 First-Order Hyperbolic Systems ... 79
4.1 First-order versions of second-order systems ... 79
4.1.1 Introducing extra first-order quantities ... 79
4.1.2 Ordering ambiguities ... 81
4.1.3 First-order Z4 system (normal coordinates) ... 82
4.1.4 Symmetry breaking: the KST system ... 83
4.2 Hyperbolic systems ... 86
4.2.1 Weak and strong hyperbolicity ... 86
4.2.2 1D Energy estimates ... 89
4.2.3 Symmetric-hyperbolic systems ... 91
4.3 Generic space coordinates ... 93
4.3.1 First-order fields ... 93
4.3.2 Generalized harmonic formulations ... 96
4.3.3 First-order Z4 formalism ... 98

4.4 Boundary conditions . . . 100
4.4.1 Algebraic boundary conditions . . . 101
4.4.2 Energy methods . . . 103
4.4.3 Robust stability test . . . 106
References . . . 108

5 Numerical Methods . . . 109
5.1 Finite difference methods . . . 110
5.1.1 Accuracy and stability . . . 110
5.1.2 The method of lines . . . 112
5.1.3 Artificial dissipation . . . 113
5.1.4 The gauge waves test-bed . . . 115
5.2 Finite volume methods . . . 117
5.2.1 Systems of balance laws . . . 118
5.2.2 Weak solutions . . . 120
5.2.3 Flux formulae . . . 123
5.2.4 High-resolution methods . . . 126
5.2.5 The modified flux approach . . . 130
5.3 Simple CFD tests . . . 132
5.3.1 Advection equation . . . 133
5.3.2 Burgers equation . . . 135
5.3.3 Euler equations: Sod test . . . 137
5.3.4 MHD equations: Orszag–Tang vortex . . . 138
References . . . 141

6 Black Hole Simulations . . . 143
6.1 Black Hole initial data . . . 143
6.1.1 Conformal metric decomposition . . . 146
6.1.2 Singular initial data: punctured black holes . . . 147
6.1.3 Regular initial data . . . 149
6.2 Dynamical time slicing . . . 155
6.2.1 Singularity avoidance . . . 155
6.2.2 Limit surfaces . . . 157
6.2.3 Gauge pathologies . . . 159
6.3 Numerical Black Hole milestones . . . 160
6.3.1 Short-term simulations . . . 160
6.3.2 Long-term simulations . . . 166
6.3.3 Further developments . . . 167
References . . . 169

7 Matter Spacetimes . . . 171
7.1 Scalar fields . . . 172
7.1.1 The Klein–Gordon equation . . . 172
7.1.2 Boson stars initial data . . . 174
7.1.3 Evolution of a single boson star . . . 178

7.2 Electromagnetic fields ... 179
7.2.1 Maxwell equations ... 180
7.2.2 Electromagnetic potential ... 181
7.2.3 The electromagnetic fields ... 183
7.2.4 The electromagnetic stress–energy tensor ... 185
7.3 Hydrodynamics ... 185
7.3.1 Perfect fluids ... 186
7.3.2 The equation of state ... 189
7.3.3 Neutron stars ... 192
7.4 Magnetohydrodynamics ... 195
7.4.1 The MHD evolution equations ... 196
7.4.2 Generalized Ohm's law ... 197
7.4.3 Ideal MHD ... 200
7.4.4 The force-free limit ... 204
7.5 Further developments ... 205
7.5.1 Boson stars collisions ... 205
7.5.2 Neutron stars collisions ... 207
References ... 209

Index ... 211