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Hydrodynamic Limits of the Boltzmann Equation

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Preface

The material published in this volume comes essentially from a course given at the Conference on "Boltzmann equation and fluidodynamic limits", held in Trieste in June 2006. The author is very grateful to Fabio Ancona and Stefano Bianchini for their invitation, and their encouragements to write these lecture notes.

The aim of this book is to present some mathematical results describing the transition from kinetic theory, and more precisely from the Boltzmann equation for perfect gases to hydrodynamics. Different fluid asymptotics will be investigated, starting always from solutions of the Boltzmann equation which are only assumed to satisfy the estimates coming from physics, namely some bounds on mass, energy and entropy. In particular the present survey does not consider convergence results requiring further regularity. However, for the sake of completeness, we will give in the first chapter some rough statements and bibliographical references for these smooth asymptotics of the Boltzmann equation, as well as for the transition from Hamiltonian systems to hydrodynamics.

Our starting point in the second chapter is some brief presentation of the Boltzmann equation, including its fundamental properties such as the formal conservations of mass, momentum and energy and the decay of entropy (for further details we refer to the book of Cercignani, Illner and Pulvirenti [31] or to the survey of Villani [106]). We then introduce the physical parameters characterizing the qualitative behaviour of the gas, and we derive formally the various hydrodynamic approximations obtained in the fast relaxation limit, i.e. when the collision process is dominating. We finally introduce the main existing mathematical frameworks dealing with the Cauchy problem for the Boltzmann equation, which can be useful for the study of hydrodynamic limits : we will particularly focus on the notion of renormalized solution defined by DiPerna and Lions [44], which will be used in all the sequel.

The third chapter is devoted to some technical results which are crucial tools for the mathematical derivation of hydrodynamic limits. Note that the general strategy to rigorously justify the formal asymptotics is to proceed by analogy, that is to recognize the structure of the expected limiting hydrodynamic model in the corresponding scaled Boltzmann equation. These tools will therefore not be equally used in all fluid regimes. The first point to be discussed is the implications of the entropy inequality, which provides some bound on the (relative) entropy, as well as some control on the entropy dissipation, and possibly some estimates on a boundary term known as the Darrozès-Guiraud information, depending on the scaling to be considered. The second point is to understand how these bounds, especially that on the entropy dissipation, allow to control the relaxation mechanism, and which consequences this implies on the distribution function. Note that, for fluctuations around a global equilibrium, such a study goes back to Hilbert [65] and Grad [59]. The last point to be investigated is the balance between this relaxation process due to collisions, and the other important physical mechanism, namely the free transport : in viscous regime the global structure of the scaled Boltzmann equation is actually of hypoelliptic type, and one can exhibit some regularizing effect of the free transport (extending for instance the velocity averaging lemma due to Golse, Lions, Perthame and Sentis [53]).

The incompressible Navier-Stokes limit, studied extensively in the fourth chapter, is therefore the only hydrodynamic asymptotics of the Boltzmann equation for which we are actually able to implement all the mathematical tools presented in Chapter 3, and for which an optimal convergence result is known. By “optimal”, we mean here that this convergence result

- holds globally in time;
- does not require any assumption on the initial velocity profile;
- does not assume any constraint on the initial thermodynamic fields;
- takes into account boundary conditions, and describes their limiting form.

We start by recalling some basic facts about the limiting system, in particular its weak stability established by Leray [70]. We then explain the general strategy used to establish the convergence result of the renormalized solutions to the suitably scaled Boltzmann equation (which is very similar to the weak compactness argument of Leray), as well as the main difficulties to be overcomed.

The moment method, introduced by Bardos, Golse and Levermore [5] requires indeed to understand how one recovers the local conservation laws in the limit, and to determine the asymptotic behaviour of the flux terms, especially of the convection terms which are quadratic functions of the moments. In order to do so, the moments are actually proved to be regular with respect to the space variables x by a refined version of the velocity averaging result due to Golse and the author [56]. Furthermore the high frequency oscillating parts of the moments, known as acoustic waves, are filtered out by a compensated compactness argument due to Lions and Masmoudi [76]. One therefore gets a global weak convergence result ([54] or [55]) which does not require a precise study of the relaxation or oscillation phenomena.

In the case of a domain with boundaries, one has further to take into account the interactions between the gas and the wall, which leads to a braking condition if the kinetic condition is a diffuse reflection, and a slipping condition if the kinetic condition is a specular reflection.

The state of the art about the incompressible Euler limit, which is the main matter of the fifh chapter, is not so complete as for the incompressible Navier-Stokes limit. Due to the lack of regularity estimates for inviscid incompressible models, the convergence results describing the incompressible Euler asymptotics of the Boltzmann equation require additional regularity assumptions on the solution to the target equations.

Furthermore, the relative entropy method leading to these stability results controls the convergence in a very strong sense, which imposes additional conditions either on the solution to the asymptotic equations ("well-prepared initial data"), or on the solutions to the scaled Boltzmann equation (namely some additional non uniform a priori estimates giving in particular the local conservation of momentum and energy).

Under these additional a priori estimates, it is indeed possible to improve the relative entropy method, so as to take into account the acoustic waves and the Knudsen layers.

The last chapter of this survey is devoted to the compressible Euler limit, and is actually a series of remarks and open problems more than a compendium of results. The main challenge is of course to understand how the entropy dissipation concentrates on shocks and discontinuities, which should be studied in one space dimension.

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Laure Saint-Raymond

Contents