Complex and Chaotic Nonlinear Dynamics

Thierry Vialar

Complex and Chaotic Nonlinear Dynamics

Advances in Economics and Finance, Mathematics and Statistics

Springer

Dr. Thierry Vialar
University of Paris X
Dept. of Economics
200, ave de la République
92001 Nanterre Cedex
France
www.isnld.com
t.vialar@isnld.com

Pr. Alain Goergen
École normale supérieure de Cachan
Dept. of Economics
61, ave du Président Wilson
94235 Cachan Cedex
France

ISBN 978-3-540-85977-2 e-ISBN 978-3-540-85978-9

Library of Congress Control Number: 2008937818

Cover design: WMXDesign GmbH, Heidelberg

Printed on acid-free paper

springer.com

To
Mr. Professor Alain GOERGEN
with my deep gratitude
and admiration.

Preface

Since the 1970s, Complex and chaotic nonlinear dynamics (in short, Complex Dynamics) constitute a growing and increasingly important area that comprises advanced research activities and strongly interdisciplinary approaches. This area is of a fundamental interest in many sciences, including Economics.

Let us start with a comment about the interest of Complex Dynamics in Economics and in so doing the necessity of such a book and its interdisciplinarity: Mathematics in Economics have a very strong didactic role. Mathematics state theoretical models and paradigms that must conform to measurements. However, in Economics the measurements are rare, more often of a small number of points and of a very low density, except for stock markets. This chasm that separates Economics from the "dense" measurable reality has maintained economists in a kind of "isolation" (compared with other sciences): (1) that of the qualitative approach that can be conducted, for instance, in a literary way, very often of great relevance but which is not quantitative, (2) and that of the construction of models ("mathematical idealities") with a strong didactic or mechanistic vocation often far from the richness of "the living". This is above all a problem of *measurement*. Beyond the epistemological revolution of nonlinear theory, today there is that of the information systems and networks, which will provide the exceptional opportunity to capture dense measurements in numerous fields of Economics (e.g. from consumer behaviors up to national accounts). Thus, the economists will be in an opposite situation than before. The measure flows for the economists will have densities increasingly similar to those of other sciences whose measures come from "the living" for example. Economics will have to treat these measure flows with relevant tools, which are necessary to master. This is a turning point for Economics. Thus, Mathematics and its analytic tools are more relevant than ever for economists, in particular to study Complex Dynamics and to bring closer theoretical models and information coming from signal or time series measurements. Calculation capabilities, networks, measurements and information treatment also make the existence of such a book legitimate and necessary.

In the same vein, Economic Policy needs tools going beyond simple observation (shifted in time) offered by statistical series in order to recognize the exact and not only the apparent state of conjuncture. This book describes these tools, with a wealth of details and precisions, and not only the tools but also many concrete applications to economic series in general. When Clive W.J. Granger published his work about the time series analysis 40 years ago, he exposed the means available at that time, of which Fourier series decomposition. These means had been refined and improved with the aim of applying them, for example, to telecommunications. The series on which telecommunications analysts work contain a great number of points, the edge effects are very often negligible and, especially, the series are almost always stationary. The use of more elaborated tools than the Fourier series decomposition is a necessity. In Economics, this had been different for a long time; everything created a problem in time series analysis, a weak extent, edge effects, a non-stationariness which cannot be reduced to the existence of a tendency, as the basic handbooks could make it believe, but is expressed by a high volatility, relative to an average value which does not have great sense, and variable according to the selected sample.

In such a context, the content of this book shows how much the recent contributions of signal theory in relation with nonlinear dynamics are powerful means of analysis and have a so important potential. Information systems and networks will contribute to this goal. In this regard, let us point out that the third part of the book is a quite essential contribution. It covers signal theory, not only in a didactic way (Fourier, Wiener, Gabor, etc.) but also by presenting highly advanced contents (polyspectra already used by economists, best basis, multiresolution analysis, hybrid waveform dictionaries, matching pursuit algorithms with time-frequency atoms, etc.). The applications are numerous and demonstrative: stock market indexes, standard signals, signals of coupled oscillators or turbulent phenomena highlighting coherent structures. Signal theory certainly has to be promoted in Economics; this book contributes to this aim.

More than in the past, Economics calls for nonlinear formalizations which provide complex formal solutions. The increasingly frequent necessity to carry out digital simulations after still largely heuristic "calibrations" leads to thorough analyses of simulated series and reference-series (often reconstructed) for which this book offers particularly adapted tools.

What appears most clearly is the innovation and originality of many parts of this book, the diversity of the applications and the richness of the theoretical exploration possibilities. This is what makes this book a document from now on impossible to disregard for economists as for econometricians, and potentially for practitioners of other disciplines.

To end this preface, may I wish that the readers have as much pleasure as I to peruse this work that numerous illustrations make less austere without ceasing to be rigorous, and then, convinced by the diversity of the applications, that the readers implement themselves the tools.

Ecole Normale Supérieure of Cachan, France

Professor Alain Goergen

Summary

Contents

Introduction

The aim of this work is to try to offer a stimulating environment for the study of complex or chaotic nonlinear Dynamics. The topicality of this type of dynamics results from widely different scientific disciplines. And although keeping an economic or financial prevalence, the assigned objective can only be approached by an opening to the other disciplines related to the subject.

Economic models have long been elaborated from constructions whose algebraic nature was of a linear order. This factor, coupled with the fact that still a few decades ago, constraints linked with calculation possibilities were strong, weighed heavily on the way of apprehending and understanding economic and scientific phenomena in general. The discovery or rediscovery, more than 30 years ago, of the different types of behavior that a simple equation of a nonlinear nature can offer, opened considerable possibilities in the formalization of economic and financial phenomena. "The great discovery of the nineteenth century was that the equations of Nature are linear, and the great discovery of the twentieth century is that they are not" (Körner 1988). This assertion which consists in saying that *the world is nonlinear* penetrates economic realities which do not escape from this observation. Even if the writing of nonlinear models precedes this rediscovery, the possibilities of simulation and experimentation are immense today. It is around this concept of *nonlinearity*, adapted to the formalization of natural phenomena and around *chaotic dynamics* that this whole book is organized. They constitute the vital leads of the four Parts of this book:

- *Part I.* The first part presents investigation methods of complex and chaotic nonlinear dynamics, among which the concepts of *nonlinear theory* (often called chaos theory) and also *nonlinear signal processing*.
- *Part II.* The second part reviews the evolution of *statistical analysis* towards nonlinear and chaotic dynamics.
- *Part III.* The third part, dedicated to *spectral and time-frequency analyses*, underlines the contributions of waveforms and atomic decompositions to the study of nonlinear phenomena.
- *Part IV.* The last part aims to depict the evolution of linear *economic growth models* towards nonlinear models, and the growing importance of nonlinearities in the construction of models.

T. Vialar, *Complex and Chaotic Nonlinear Dynamics*,

Part I

Since the 1970s the irruption of the "nonlinear" led to a profound transformation of numerous scientific and technical fields. Economics does not escape this revolution. The taking into account of *nonlinearities is an infinite source of behavior diversity*, that makes it possible to better understand *natural phenomena* and phenomena considered as complex which were adverse to any modeling before. It is indeed a true *epistemological rupture*, that occurred in 1971,[1] with the introduction of the concepts of "deterministic chaos", *sensitive dependence on initial conditions* and dissipative systems. Henceforth, we understand how *an apparent disorder can dissimulate a subjacent order*. Thus, it appears important to assemble the *theory core* and the *tools* to *investigate* these *complex* and *nonlinear dynamics*, which escaped any analytic effort before. These investigation methods are not necessarily recent. Indeed, they can have distant origins in time, but today it is possible to speak of a "modern conceptual unification" through the notions of sensitive dependence on initial conditions, bifurcations,[2] subharmonic cascades, attractors, Lyapunov exponent, saddle-connection, transitions to chaos, dissipative systems, conservative systems, hyperbolicity, hyperbolic systems, etc. Although we are faced with a coherent set, this set is only at an early stage.

Inside this set appear techniques developed within the last 20 years, gathered together under the name "nonlinear signal processing" based mainly on the Takens theorem. It is a theorem of time series reconstruction, based on the concept of *topological equivalence*, which enables to identify the nonlinear nature of an original time series, for example, periodic, quasiperiodic, aperiodic or chaotic, while saving a huge amount of calculation. The stake here is extremely high. Indeed, without necessarily knowing the equations of the dynamical system which generated a series and by working in a *reconstructed phase space* of a very low dimension, it is possible to reproduce the essential features of a system or an original trajectory. The study of the *geometrical objects of low dimension can provide all the information* which we need. In Economics very often we have to face this type of problem where we do not necessarily know the number of variables involved in dynamics nor the dimension of the system which can be infinitely large. However, it is known that large attractors or infinite systems can have low dimensions. Thus, with largely reduced series, the *study of low-dimension objects can reveal the information* that we need to identify dynamics. It is possible to consider this as a diagnosis method, like the Poincaré map. This approach leads directly to a major concept which is that of the capacity dimension,[3] which is a non-integer dimension. This concept is an instrument quite as important as the Lyapunov exponent. On one hand, it makes it possible

[1] But the origin dates back Lorenz in the 1960s, where new mathematics were born.

[2] In connection with these tools, it seemed important to highlight the symptomatic phenomenon which links the *speed* of transition between two states of a dynamical system with the characteristics of the periodic or chaotic regime of the final state of the system. The section is entitled the *bifurcation paradox* exhibiting the title of the authors of the study.

[3] Also called Kolmogorov dimension, or "box counting", also named Hausdorff dimension; the whole being regrouped under the name of fractal dimension.

to characterize the attractor that we have to face and, on the other hand, to make *the difference between deterministic chaos* and *random walk*. Generally, it is said for example that a "Brownian motion" has a capacity dimension equal to two, which is not necessarily the case of an apparent deterministic chaos.[4] It is the concept of stability of a dynamics, approached in particular through the *Floquet theory*, which leads us to define the notion of *topological invariant set* which then leads to define the *attractor* concept.[5] An attractor can be known as a fixed-point, limit-cycle, toric or strange attractor. It is characterized by its capacity dimension, as well as the Euclidean dimension of the system in which the attractor appears, knowing that the latter can have only a non-integer dimension per definition. A system which does not have an attraction area in the phase space is known as *conservative*, in the reverse case it is known as *dissipative*. Thus, the existence of an attractor type in a dynamics characterizes *dissipative systems*. It will be noticed however that *chaotic or very complex behaviors can singularly exist in models considered conservative*. This is indeed the case in systems of the *Hamiltonian type*, as it is possible in Economics.[6]

Like the methods mentioned above, which are assembled under the name of "nonlinear signal processing", there is a major tool which will be also largely developed from another aspect in the part III, which is *spectral analysis* or power spectrum of a dynamics. Spectral analysis results from two basic concepts which are the Fourier transform and the autocorrelation function of a signal, which is an extraction of the Wiener–Khintchine theorem. *The temporal autocorrelation function* measures the resemblance or the similarity of values of a variable in time. In fact *this temporal correlation function corresponds to the Fourier transform of the power spectrum*. For chaotic regimes, for example, the similarity decreases with time. It is said that they are unpredictable due to the loss of internal similarity to their processes. When we have a time series, an essential task is to determine the type of dynamics which engendered it, be it a more or less complex oscillation but of a defined period, or a superposition of several different oscillations, or other types of dynamics. The periodicities are identified with the spectral analysis method, whether the subjacent model is known or not. This already evoked phenomenon, highlights the fact that certain dynamics result from a superposition of oscillations with different amplitudes and oscillations but also from harmonics of these oscillations. In the last case, the regime which is described as quasiperiodic has an associated attractor which is of a higher order than the limit-cycle, i.e. a torus for example. For chaotic but deterministic regimes, i.e. for dynamics represented by a limited number of nonlinear differential equations, the attractor in the phase space is a strange attractor. Thus, this method of spectral analysis makes it possible to identify the nature of a dynamics, periodical, aperiodic or quasiperiodic. Besides, we experiment this technique on various types of signals resulting from the

[4] See section about "(non-fractional) Brownian motions".

[5] We sometimes find an unsuitable terminology to characterize an attractor and in particular the term of "chaotic attractor", which in fact is sometimes used to indicate an attractor which exists in a chaotic regime.

[6] Refer for cyclic growth models in Economics to Goodwin (1967).

logistic equation, from ARMA processes, from stock market courses as the French stock-market index (Cac40), or from the Van der Pol oscillator.

The logistic model symbolizes the paradigm of a nonlinear model. Thus, it is used transversely to highlight numerous notions quoted previously concerning nonlinearities. It is also used to experiment the delay-model applied to the logistic equation which introduces by convolution a discrete delay into the construction of an economic model. The lengths of the delays are distributed in a random way in the population. The delay is in fact modelled by means of a random variable which is characterized by its probability distribution. It will be noted that such a system topples more tardily in the chaos, i.e. there is a shift of the bifurcation points but also an unhooking in the trajectories.

The *singular spectrum analysis* (SSA) method is the last investigation method concerning the complex dynamics presented in this part. The method associates the Takens reconstruction technique and the technique called the *singular value decomposition* in matrix algebra.[7] If we simplify, the method consists in the projection of a time series on a basis of eigenvectors extracted from this same time-series. In fact, a *matrix trajectory* is projected on the space described by the eigenvectors of the time series covariance matrix. The eigenvalues obtained can be ordered and be filtered for the extraction of the *deterministic* part of the signal cleaned of its background noise.

Part II

This part aims to depict the evolution of statistical analysis towards nonlinear stochastic processes and chaotic processes. The purpose is to state the main developments concerning this subject: (a) End of the domination of the ARMA model, (b) Nonlinearity Tests (BDS), (c) Stakes of the non-parametric analysis, (d) A statistical theory of chaotic dynamics is to be built, (e) Long-memory process and self-similarity, (f) Construction of ARFIMA models, (g) FIGARCH models and volatility of variances, (h) Lo and MacKinlay tests about the rejection of the random-walk hypothesis for stock-exchange markets, (i) Estimations of the Hurst exponent (in particular by means of the Abry–Veitch wavelet method), (j) Estimators of density, (k) Invariant measurement of a dynamical system and (limit) invariant density,[8] (l) Ergodic theory, etc.

ARMA modeling is representative of linear modeling, but the linearity Hypothesis is unsuitable to represent real phenomena in many fields. The nonlinear economic models introduced in 1950 and 1951 by Hicks and Goodwin, because of the absence of statistical tools, did not have the deserved resonance that they should have had at that time. This explains the long domination of the ARMA models until the nonlinearity Hypothesis found its own consistency, in particular in statistics.

---

[7] Karhunen-Loève method.

[8] *Invariant density:* Invariant density is also called "natural" invariant or "natural" density.

Thus in 1976, the first linearity Test has been developed by Granger and Newbold. More recently, the non-parametric Test Statistic (BDS test) was taken as a nonlinearity test. It helps to provide information about nonlinearity, however it is not a measurement of the nature of the time series chaos, but in this part it is used as an introduction to non-parametric statistical analysis. Parametric analysis aims to rebuild a deterministic model subjacent to a time series. The rebuilding is done by means of a stochastic model with the implicit idea of the existence of a subjacent structure. The stochastic parametric modeling of nonlinear processes does unfortunately not make it possible for a great number of processes to produce robust estimates. In front of this lack of specification about the complex or chaotic processes, our attention is drawn to *non-parametric analysis*, which does not seek to specify models, but generally aims to rebuild trajectories without a deterministic or stochastic model. The method by means of the extraction of the time-series properties and its estimators allows the reconstruction of the dynamic relation which links the time-series terms. It is said that the dynamic relation is estimated in a non-parametric way.

Faced with these complex or chaotic dynamics, Chaos theory provides chaos detection tests, as the Lyapunov test and the correlation dimension, but they are not statistical tests in a strict sense. Thus, the statisticians have been led to elaborate statistical validation tests of these detection tests; in particular the *random mixture* test. The method is surprising and the idea is that the mixture destroys the possible deterministic structure of a time series. If the mixed series loses its structure, the correlation dimension and the Lyapunov exponent of the mixture must make it possible to distinguish it. In spite of encouraging results, contradictions between the results of Lyapunov and the correlation dimension prevent convincing conclusions about the deterministic or non-deterministic nature of the observed chaotic dynamics. In the light of this type of example, it seems fundamental to build a statistical theory core of chaotic dynamics, which still remains to be worked out in spite of numerous current contributions.

A way of characterizing the nature of chaotic dynamics is to study the structure of long-memory processes. These processes are observed in numerous fields, for example in the telecommunications sector in connection with the information flow on the Internet, but also in connection with financial markets. They are detected by the observation of their autocorrelation functions, which decrease hyperbolically towards zero, whereas they decrease exponentially for short-memory processes. (The hyperbolic decrease can also express a nonstationariness.) The long memory is also detected by spectral concentrations which increase when we approach the central frequency centered at zero, or by a persistent or anti-persistent behavior. It is commonly said that the more a process is persisting, the more the convergence is slow and the more the sum of the autocorrelations is high. For a process with short memory, the sum of the autocorrelations is weak. In short, we are interested in the speed of the hyperbolic and geometrical convergence towards zero. Weak lags and strong correlations rather characterize models with short memory of the ARMA type. The long memory processes lead us to outline the ARFIMA processes which integrate the long memory phenomena (Long Range Dependence: LRD).

The Hurst exponent allows to introduce long memory into an artificially generated process. It is the purpose of the numerical generators of fractional Brownian motions. The parameter of fractional integration, i.e. unit root, is used in ARIMA processes to test stationarity. Moreover, a functional relation was highlighted between this parameter and the Hurst exponent, consequently assigning a new role to it. This role is to introduce long-term dependence (or long memory) into new models, i.e. the ARFIMA processes.[9] Recently, the estimate methods of the Hurst exponent have been developed in empirical series.[10] And the effectiveness of these methods can be tested with a good safeness, since it is possible to estimate a parameter that we have fixed before (a priori) to construct an experimental series. We can proceed in a similar way with an ARFIMA process. Among the estimation methods of the Hurst exponent, the technique developed in 1998 by Abry and Veitch using the wavelets properties appears to be important, to which we will give a particular place.

The ARCH processes supplanted the ARMA processes, unsuited to financial series which have asymmetrical structures and strong volatility of variance. ARCH processes integrate the parameters of the conditional variance in an endogenous way and have often been used in the optimization of (financial) portfolio choices. The study of the conditional variance makes it possible to highlight the persistence of shocks, by using an extension of IGARCH processes (integrated GARCH), i.e. FIGARCH processes (Fractionally Integrated GARCH).

Closely related to the birth of probability theory, the random walk hypothesis has a famous history, whose actors are Bachelier, Lévy, Kolmogorov and Wiener. More recently, one of the first applications of the random-walk hypothesis to the financial markets dates back to Paul Samuelson in 1965, whose contribution has been developed in an article entitled "Proof that Properly Anticipated Prices Fluctuate Randomly". He explains why in an efficient market, concerning information, the price changes are unpredictable if they are properly anticipated, i.e. if they fully incorporate the expectations, information and forecasts of all the market participants. In 1970, Fama summarizes what precedes in a rather explicit formula: "the prices fully reflect all available information". Contrary to numerous applications of the random walk hypothesis in natural phenomena, for which the randomness is supposed almost by default due to the absence of any natural alternative, Samuelson argues that the randomness is achieved through the active participation of many investors who seek increase of their wealth. They attempt to take advantage of the smallest information at their disposal. And while doing so, they incorporate their information into the market prices and quickly eliminate the capital-gain and profit opportunities. If we imagine an "ideal" market without friction and trading-cost, then the prices must always reflect all information available and no profits can be garnered from the trading based on information *because "such profits have already been captured"*. Thus in a contradictory way, the more the market is efficient, the more the price time-series generated by such a market is random, and "the most

[9] Hosking in 1981 and Granger, Joyeux in 1980.

[10] Some of them use the concept of Embedding space resulting from the Takens theorem.

efficient of markets is one in which the price changes are completely random and unpredictable". Thus the random walk hypothesis and the efficient markets hypothesis became emblematic in Economics and Finance, although more recently, in 1980, Grossman and Stiglitz considered that the efficient market assumption is an economically unrealizable idealization. Moreover, some recent works done during the last 15 years initiated an approach aiming to reject the random walk hypothesis. Econometric studies conducted by Lo and MacKinlay (since 1988) relating to the US stock-exchange market rejected the random walk hypothesis for weekly values of the courses of the New York Stock Exchange, using a simple test based on the volatility of the courses. They specified however that the rejection of the random walk assumption does not necessarily imply the inefficiency of the stock-price formation. We will outline this test to show how the academic assumption of random walk for the financial markets is subjected to critiques from statisticians today.

A way of approaching the study of statistical properties of *chaotic processes* results from the *Birkhoff and Von Neumann works* about the *invariant distributions*[11] (distributions which have a positive Lebesgue measure). Dynamics of *aperiodic nature* sometimes have variables with distributions of this type, *which indicate the frequency with which they take values in a given interval*. The most analyzed invariant distributions are those which can be represented by a density function. Techniques have been developed in order to build such functions for chaotic processes.

Part III

One of the characteristics of behaviors belonging to nonlinear models is to highlight transitory phenomena, intermittencies, turbulences or chaotic dynamics. They correspond to a tool which allows the representation of phenomena of the stationary types already depicted by linear models, and, also at the same time, phenomena of the periodic and turbulent or chaotic type. This faculty of representation is particularly useful. The complexity of the phenomena observed empirically thus finds an algebraic tool which makes it possible to depict this complexity. The writing of dynamical systems must be able to *depict the coexistence of simple and complex solutions*, by the elementary play of parameter setting. Apparently, deterministic chaos resulting from nonlinear models does not seem different from a random walk measured on a natural phenomenon, a stock-exchange or economic phenomena. However, in the first case, we are the "holders" of the equations of the system, whereas in the second case, we do not know if these equations exist.

This evolution of the "linear" towards the "nonlinear" can find a symbolic illustration in the overtaking (which proved to be necessary) of the Fourier analysis of certain natural phenomena. In his study about Heat in the nineteenth century, Fourier showed how his works help to understand the Natural phenomena by helping to

[11] Absolutely continuous invariant distributions which have a positive Lebesgue measure.

numerically solve the equations which hitherto were refractory. For a number of differential equations, the Fourier transformation replaces a complicated equation by a series of simple equations. Each one of these equations indicates the temporal evolution of the coefficient of one of the sinusoids composing the initial function. Each coefficient has its own equation. Thus the Fourier transform had an important success, insomuch it was often used for problems to which it was unsuited (Meyer 1992, p. 86). The Fourier analysis is not appropriate for all signals nor for all problems. In fact, "the Fourier analysis helps to solve *linear problems, for which the effect is proportional to the cause*" (Hubbard 1998). Nonlinear problems are more difficult to solve. Generally, the interacting variables form systems whose behaviors are unstable. And faced with this type of problem we treat it "as if it was linear". Indeed, certain complex natural phenomena that would require the use of nonlinear partial differential equations are not solved in this manner because they cannot be solved. It is by reducing the difficulty to a linear equation, which can be solved by the Fourier analysis, that answers were brought. In addition, this is a reality that the economists had to face in the past. The elements of the Fourier analysis are sines and cosines which oscillate infinitely with a fixed frequency. And "in this context of infinite time, the following expression, *changing frequency,* becomes a contradiction".[12] "The Fourier analysis is thus unsuited to the signals which change abruptly and in an unpredictable way" (Hubbard 1998), and however these events contain the most information. Now, "wavelet analysis is a manner of expressing our sensitivity to the variations".[13] Thus, the abrupt and unforeseeable variations are "read" by the wavelets. The study of turbulence phenomena, i.e. sudden unpredictable chaotic variations which are representative of nonlinear problems, are in this respect symbolic of the efficiency lack of the previous investigation methods.

The Fourier analysis is not appropriate to the nonlinear problems that we meet in turbulence phenomena, the nature of the wavelets is more adapted. The turbulence appears on scales of very diverse frequencies and the wavelets are also adapted to the analysis of the interactions between scales. In spite of what precedes, there is no ideal tool to resolve nonlinear behaviors. All of these observations about waveform analysis is one of the objects of the third part of this work.

The time-frequency theory offers transformation methods of time-series. A complete statistical theory about time-frequency analysis does not exist yet, in spite of certain recent work developed from the starting point of the Wiener deterministic spectral theory. Without covering this vast and very difficult topic here, a very particular interest however is granted to the statistical and econometric properties of the wavelet transform of stock-exchange and economic time series. We will argue in favor of wavelets and their particular properties, their adaptation to nonstationarities and to abrupt variations of signals, as it was evoked previously. Most of all, we will highlight the role of *atomic decompositions* of signals which use *time-frequency atoms* (i.e. *waveforms dictionaries*, which contain different types of waveforms to decompose a signal). *Numerical imaging* in time-frequency planes

[12] D. Gabor quotation, Nobel Prize of Physics 1971 for the invention of holography. Recovery in the work of Hubbard (1998).

[13] Yves Meyer, one of the founders of the wavelet theory.

(or in three-dimensional time-frequency-amplitude spaces) are also a considerable contribution to the comprehension of the subjacent (or hidden) structures for signals whose origin is natural, financial or economic. A contribution for which it is probably necessary, in economic and financial matters, to develop reference frameworks, i.e. for example data banks of *imagings* and *forms* of *structures* calibrated on academic signals. This could be the case for signals whose origin is the stock-exchange that we commonly consider as following a "random walk".

Before all new analysis methods briefly mentioned previously, to which it is advisable to add *polyspectral analysis*, the classical "spectral analysis" *has been already experimented in* Economics in the past. Indeed, the first work dates back to Kendall in 1922, spectral analysis has been experimented on the Beveridge corn price index. More recently, it is possible to quote the fundamental contribution of C.W. Granger in 1969, those of M.W. Watson in 1993 (Watson 1993), but also the work of Wen Y. in 1998 (Wen 1998), applied to the Real Business Cycle theory, as those of Perli and Sakellaris (1998) about the same topic. We previously evoked the fact that, apparently, the deterministic chaos resulting from nonlinear models does not seem different from a random walk measured on a stock-exchange market for example. However for the first, we know the equations, whereas for the second we do not know if they exist.

The recent Lo and MacKinlay results mentioned in the preceding part, which reject the random walk hypothesis about stock-exchange markets, can be related to the conclusions of quite recent work (although of very different nature) presented by J.B. Ramsey and Z. Zhang about the Standard & Poor's stock 500 index;[14] Thus, we are interested in describing the "Matching Pursuit Accord" (Mallat and Zhang 1993) algorithm with its dictionaries of "waveforms" (i.e. Gabor or time-frequency atoms) conceived by S. Mallat and Z. Zhang, and that we furthermore applied to the French stock-exchange index (cac40). Meeting the statistical Lo and MacKinlay analysis, the decomposition (into "time-frequency atoms") of a signal could make it possible to discriminate its random or non-random characteristic. The argument is based on the fact that the number of waveform structures necessary for the decomposition of a "natural" random series is higher than the number of waveform structures used to decompose a stock-exchange series. Such an analysis would indeed plead in favor of the idea that stock-exchange markets do not follow a random-walk, contrary to the "traditional" conclusions of statistical analysis. Moreover, within this same framework, a detailed attention was given to the treatment of *internal or external shocks of a signal*, in particular concerning the Dirac (delta) function. Indeed, this same work attempts to show that the high-energy bursts of a time series which involve *almost all* frequencies of a spectrum would make it possible *to discriminate* the internal or external origin of shocks of the aforementioned time series.

[14] The S&P500 is decomposed through the "Matching Pursuit Accord" by means of dictionaries of time-frequency atoms.

Part IV

The last part aims to depict the growing place of nonlinearities in the conception of economic growth models. The models used during the twentieth century were primarily of linear nature, including those aiming to represent business cycles; in particular we think about the Frisch–Slutsky linear model (1930). The cycles resulted then from the propagation of external shocks in our economies and were not generated in an endogenous way by the model. All the models deriving from these principles quickly showed their limits. The use of nonlinear models proved to be necessary, Hicks in 1950 and Goodwin in 1951 had already introduced them, but they had at that time a weak resonance, amongst other reasons due to the lack of statistical and computational tools. Note that the first application of "catastrophe theory" to economics, and in particular to Kaldor model (1940) seems to date back to H.R. Varian (1979). Nevertheless, the linearity hypothesis is so simple to pose and exploit, that is why historically it has long been used.

The *AK* models for example concern this hypothesis, where the output is constantly proportional to the input and *A* is taken as a parameter. Endogenous growth is generated because of a linearity in the differential equation of the model. However, this hypothesis is subject to critiques and is not necessarily relevant. Indeed, on the one hand, the basic rules of the economic theory, which state the decrease of marginal productivity or marginal utility, implicitly imply that the relation between input and output is of rather nonlinear nature. In this respect, the Von Neumann model, based on relations of linear nature, avoids this pitfall, since it is not built from the neo-classical production function.

Moreover, even within the framework of models whose construction is linear, the nature of the domain of definition in particular at its boundaries, can be the source of nonlinearities because of "edge-effects". In this respect, we can evoke the variables which are rates and ratios, like the per capita variables for example. The "nonlinearity" hypothesis is more relevant than its opposite which, beyond its *didactic qualities*, must give way to more realistic concepts concerning the notion of nonlinearity. As said hereinbefore, certain endogenous growth models are built from linearities, but the endogenous growth phenomenon can appear without linearity, for example from a model with two differential equations whose respective convexities of trajectories are different. Thus, in the absence of linearity, certain configurations allow the observation of endogenous growth. *The economic models gradually endogenize the mechanisms of growth*. Exogenous in the various Solow models, the factors of Growth integrated (directly or indirectly) the heart of the models, whether they result from internal mechanisms of choice optimization between consumption and savings, or from positive externalities as in the Romer model.

The conception of optimal growth models highlights a system of differential equations of order two, which determines the optimal trajectory of consumption and investment. The solutions of this type of model generally appear as a saddle-point (i.e. saddle path) which is a kind of instability basin, in which the trajectory of steady balance is narrow and depends on the initial conditions. In economic models, this type of instability basin represented by the saddle-point is symptomatic of the prevalence of unstable trajectories where equilibrium and stability are understood as *singularities*. The optimal growth model, applied to the strategy of portfolio choice, leads to the same observation and suggests a way to explain the instability of stock markets. In the Boldrin–Woodford model (1990), the optimal growth exhibits cyclic or chaotic endogeneous fluctuations. The axis of the saddle, which represents the stable trajectory, converges towards an equilibrium point, which becomes (under some conditions) a limit-cycle at the equilibrium in the Benhabib–Nishimura optimal growth model. This analysis is developed through the "Rational Expectations" school and explains the cycles (and fluctuations) of economy under a new light. The model replaces the academic notion of shock by internal behaviors of optimization which become the source of the Equilibrium cycles. An explanation of an endogenous type thus leads to a better understanding of these phenomena, but this requires to accept, as a preliminary, the probable preeminence of nonlinearities in the construction of the economic models. The hypothesis of nonlinearities, more plausible for the comprehension of natural and economic phenomena, allows to explain equilibrium cycles but also of chaotic behaviors. The models based on nonlinear algebraic structures, such as those of Day and Day–Lin, also depict these periodic or chaotic behaviors. The structure of overlapping generations models developed for instance by Benhabib–Day (1982) and Grandmont (1985) also allows to produce cyclic or chaotic dynamics.

The economic growth models attempt to depict the evolution of national income. However, stock markets, which are known as perfect markets and also known as advanced indicators of economic activity, exhibit trajectories very different from those of the national product. The characterization of these growth disparities (i.e. gaps) is not an easy task, but can be approached from various angles through the concepts of Market and Economic *Value-Creation*,[15] through the rational expectation concept and that of rational bubbles, but also through the contribution of the sunspots models.

[15] e.g., indicators such as *Market Value Added* (MVA) and *Economic Value Added* (EVA).