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Rufus Bowen

Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms

Second, revised edition

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Editor

Preface by David Ruelle

Springer

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Preface

The Greek and Roman gods, supposedly, resented those mortals endowed with superlative gifts and happiness, and punished them. The life and achievements of Rufus Bowen (1947–1978) remind us of this belief of the ancients. When Rufus died unexpectedly, at age thirty-one, from brain hemorrhage, he was a very happy and successful man. He had great charm, that he did not misuse, and superlative mathematical talent. His mathematical legacy is important, and will not be forgotten, but one wonders what he would have achieved if he had lived longer. Bowen chose to be simple rather than brilliant. This was the hard choice, especially in a messy subject like smooth dynamics in which he worked. Simplicity had also been the style of Steve Smale, from whom Bowen learned dynamical systems theory.

Rufus Bowen has left us a masterpiece of mathematical exposition: the slim volume *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms* (Springer Lecture Notes in Mathematics **470** (1975)). Here a number of results which were new at the time are presented in such a clear and lucid style that Bowen's monograph immediately became a classic. More than thirty years later, many new results have been proved in this area, but the volume is as useful as ever because it remains the best introduction to the basics of the ergodic theory of hyperbolic systems.

The area discussed by Bowen came into existence through the merging of two apparently unrelated theories. One theory was equilibrium statistical mechanics, and specifically the theory of states of infinite systems (Gibbs states, equilibrium states, and their relations as discussed by R.L. Dobrushin, O.E. Lanford, and D. Ruelle). The other theory was that of hyperbolic smooth dynamical systems, with the major contributions of D.V. Anosov and S. Smale. The two theories came into contact when Ya.G. Sinai introduced Markov partitions and symbolic dynamics for Anosov diffeomorphisms. This allowed the poweful techniques and results of statistical mechanics to be applied to smooth dynamics, an extraordinary development in which Rufus Bowen played a major role. Some of Bowen's ideas were as follows. First, only one-dimensional statistical mechanics is discussed: this is a richer theory, which yields what is needed for applications to dynamical systems, and makes use of the powerful analytic tool of transfer operators. Second, Smale's Axiom A dynamical systems are studied rather than the less general Anosov systems. Third, Sinai's Markov partitions are reworked to apply to Axiom A systems and their construction is simplified by the use of *shadowing*. The combination of simplifications and generalizations just outlined led to Bowen's concise and lucid monograph. This text has not aged since it was written and its beauty is as striking as when it was first published in 1975.

Jean-René Chazottes has had the idea to make Bowen's monograph more easily available by retyping it. He has scrupulously respected the original text and notation, but corrected a number of typos and made a few other minor corrections, in particular in the bibliography, to improve usefulness and readability. In his enterprise he has been helped by Jerôme Buzzi, Pierre Collet, and Gerhard Keller. For this work of love all of them deserve our warmest thanks.

Bures sur Yvette, May 2007 *David Ruelle*

Note of the editor. The conference "Current Trends in Dynamical Systems and the Mathematical Legacy of Rufus Bowen" (University of British Columbia, July 30, 2017 - August 4, 2017) prompted us to correct some typos and mistakes left in the second revised edition (2008). In this task, we were helped by Brian Marcus and Benoît Dagallier, and we are grateful to them for that.

Orsay, May 2017 *Jean-René Chazottes*

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These notes came out of a course given at the University of Minnesota and were revised while the author was on a Sloan Fellowship.

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Introduction

The main purpose of these notes is to present the ergodic theory of Anosov and Axiom A diffeomorphisms. These diffeomorphisms have a complicated orbit structure that is perhaps best understood by relating them topologically and measure theoretically to shift spaces. This idea of studying the same example from different viewpoints is of course how the subjects of topological dynamics and ergodic theory arose from mechanics. Here these subjects return to help us understand differentiable systems.

These notes are divided into four chapters. First we study the statistical properties of Gibbs measures. These measures on shift spaces arise in modern statistical mechanics; they interest us because they solve the problem of determining an invariant measure when you know it approximately in a certain sense. The Gibbs measures also satisfy a variational principle. This principle is important because it makes no reference to the shift character of the underlying space. Through this one is led to develop a "thermodynamic formalism" on compact spaces; this is carried out in chapter two. In the third chapter Axiom A diffeomorphisms are introduced and their symbolic dynamics constructed: this states how they are related to shift spaces. In the final chapter this symbolic dynamics is applied to the ergodic theory of Axiom A diffeomorphisms.

The material of these notes is taken from the work of many people. I have attempted to give the main references at the end of each chapter, but no doubt some are missing. On the whole these notes owe most to D. Ruelle and Ya. Sinai.

To start, recall that $(X, \mathscr{B}, \mu)$ is a *probability space* if $\mathscr{B}$ is a σ-field of subsets of X and μ is a nonnegative measure on $\mathscr{B}$ with $\mu(X) = 1$. By an *automorphism* we mean a bijection $T : X \to X$ for which

$$\text{(i)} \quad E \in \mathscr{B} \text{ iff } T^{-1}E \in \mathscr{B},$$

$$\text{(ii)} \quad \mu(T^{-1}E) = \mu(E) \text{ for } E \in \mathscr{B}.$$

If $T : X \to X$ is a homeomorphism of a compact metric space, a natural σ-field $\mathscr{B}$ is the family of Borel sets. A probability measure on this σ-field is called a *Borel* probability measure. Let $\mathcal{M}(X)$ be the set of Borel probability measures on X and $\mathcal{M}_T(X)$ the subset of invariant ones, *i.e.*, $\mu \in \mathcal{M}_T(X)$ if $\mu(T^{-1}E) = \mu(E)$ for all Borel sets E. For any $\mu \in \mathcal{M}(X)$ one can define $T^*\mu \in \mathcal{M}(X)$ by $T^*\mu(E) = \mu(T^{-1}E)$.

Remember that the real-valued continuous functions $\mathscr{C}(X)$ on the compact metric space X form a Banach space under $\|f\| = \max_{x \in X} |f(x)|$. The weak $*$-topology on the space $\mathscr{C}(X)^*$ of continuous linear functionals $\alpha : \mathscr{C}(X) \to \mathbb{R}$ is generated by sets of the form

$$U(f, \varepsilon, \alpha_0) = \left\{\alpha \in \mathscr{C}(X)^* : |\alpha(f) - \alpha_0(f)| < \varepsilon\right\}$$

with $f \in \mathscr{C}(X)$, $\varepsilon > 0$, $\alpha_0 \in \mathscr{C}(X)^*$.

Riesz Representation. For each $\mu \in \mathcal{M}(X)$ define $\alpha_\mu \in \mathscr{C}(X)^*$ by $\alpha_\mu(f) = \int f d\mu$. Then $\mu \leftrightarrow \alpha_\mu$ is a bijection between $\mathcal{M}(X)$ and

$$\left\{\alpha \in \mathscr{C}(X)^* : \alpha(1) = 1 \text{ and } \alpha(f) \geq 0 \text{ whenever } f \geq 0\right\}.$$

We identify α_μ with μ, often writing μ when we mean $\alpha(\mu)$. The weak $*$-topology on $\mathscr{C}(X)^*$ carries over by this identification to a topology on $\mathcal{M}(X)$ (called the weak topology).

Proposition. *$\mathcal{M}(X)$ is a compact convex metrizable space.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a dense subset of $\mathscr{C}(X)$. The reader may check that the weak topology on $\mathcal{M}(X)$ is equivalent to the one defined by the metric

$$d(\mu, \mu') = \sum_{n=1}^{\infty} 2^{-n} \, \|f_n\|^{-1} \, \left| \int f_n \, \mathrm{d}\mu - \int f_n \, \mathrm{d}\mu' \right| . \quad \square$$

Proposition. *$\mathcal{M}_T(X)$ is a nonempty closed subset of $\mathcal{M}(X)$.*

Proof. Check that $T^* : \mathcal{M}(X) \to \mathcal{M}(X)$ is a homeomorphism and note that $\mathcal{M}_T(X) = \{\mu \in \mathcal{M}(X) : T^*\mu = \mu\}$. Pick $\mu \in \mathcal{M}(X)$ and let $\mu_n = \frac{1}{n}(\mu + T^*\mu + \cdots + (T^*)^{n-1}\mu)$. Choose a subsequence μ_{n_k} converging to $\mu' \in \mathcal{M}(X)$. Then $\mu' \in \mathcal{M}_T(X)$. $\square$

Proposition. *$\mu \in \mathcal{M}_T(X)$ if and only if*

$$\int (f \circ T) \, \mathrm{d}\mu = \int f \mathrm{d}\mu \quad \text{for all } f \in \mathscr{C}(X).$$

Proof. This is just what the Riesz Representation Theorem says about the statement $T^*\mu = \mu$. $\square$