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Rational Algebraic Curves

A Computer Algebra Approach

With 24 Figures and 2 Tables



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Preface

Algebraic curves and surfaces are an old topic of geometric and algebraic investigation. They have found applications for instance in ancient and modern architectural designs, in number theoretic problems, in models of biological shapes, in error-correcting codes, and in cryptographic algorithms. Recently they have gained additional practical importance as central objects in computer-aided geometric design. Modern airplanes, cars, and household appliances would be unthinkable without the computational manipulation of algebraic curves and surfaces. Algebraic curves and surfaces combine fascinating mathematical beauty with challenging computational complexity and wide spread practical applicability.

In this book we treat only algebraic curves, although many of the results and methods can be and in fact have been generalized to surfaces. Being the solution loci of algebraic, i.e., polynomial, equations in two variables, plane algebraic curves are well suited for being investigated with symbolic computer algebra methods. This is exactly the approach we take in our book. We apply algorithms from computer algebra to the analysis, and manipulation of algebraic curves. To a large extent this amounts to being able to represent these algebraic curves in different ways, such as implicitly by defining polynomials, parametrically by rational functions, or locally parametrically by power series expansions around a point. All these representations have their individual advantages; an implicit representation lets us decide easily whether a given point actually lies on a given curve, a parametric representation allows us to generate points of a given curve over the desired coordinate fields, and with the help of a power series expansion we can for instance overcome the numerical problems of tracing a curve through a singularity.

The central problem in this book is the determination of rational parametrizability of a curve, and, in case it exists, the computation of a good rational parametrization. This amounts to determining the genus of a curve, i.e., its complete singularity structure, computing regular points of the curve in small coordinate fields, and constructing linear systems of curves with prescribed intersection multiplicities. Various optimality criteria for rational

parametrizations of algebraic curves are discussed. We also point to some applications of these techniques in computer aided geometric design. Many of the symbolic algorithmic methods described in our book are implemented in the program system CASA, which is based on the computer algebra system Maple.

Our book is mainly intended for graduate students specializing in constructive algebraic curve geometry. We hope that researchers wanting to get a quick overview of what can be done with algebraic curves in terms of symbolic algebraic computation will also find this book helpful.

This book is the result of several years of research of the authors in the topic, and in consequence some parts of it are based on previous research published in journal papers, surveys, and conference proceedings (see [ReS97a], [Sen02], [Sen04], [SeW91], [SeW97], [SeW99], [SeW01a], [SeW01b]).

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Contents

1	Introduction and Motivation	1
1.1	Intersection of Curves	4
1.2	Generating Points on a Curve	5
1.3	Solving Diophantine Equations	6
1.4	Computing the General Solution of First-Order Ordinary Differential Equations	8
1.5	Applications in CAGD	9
2	Plane Algebraic Curves	15
2.1	Basic Notions	15
2.1.1	Affine Plane Curves	16
2.1.2	Projective Plane Curves	19
2.2	Polynomial and Rational Functions	24
2.2.1	Coordinate Rings and Polynomial Functions	24
2.2.2	Polynomial Mappings	26
2.2.3	Rational Functions and Local Rings	28
2.2.4	Degree of a Rational Mapping	32
2.3	Intersection of Curves	34
2.4	Linear Systems of Curves	41
2.5	Local Parametrizations and Puiseux Series	50
2.5.1	Power Series, Places, and Branches	51
2.5.2	Puiseux's Theorem and the Newton Polygon Method ..	55
2.5.3	Rational Newton Polygon Method	61
	Exercises	62
3	The Genus of a Curve	67
3.1	Divisor Spaces and Genus	67
3.2	Computation of the Genus	69
3.3	Symbolic Computation of the Genus	78
	Exercises	85

4	Rational Parametrization	87
4.1	Rational Curves and Parametrizations	88
4.2	Proper Parametrizations	95
4.3	Tracing Index	100
4.3.1	Computation of the Index of a Parametrization	101
4.3.2	Tracing Index Under Reparametrizations	104
4.4	Inversion of Proper Parametrizations	105
4.5	Implicitization	108
4.6	Parametrization by Lines	114
4.6.1	Parametrization of Conics	114
4.6.2	Parametrization of Curves with a Point of High Multiplicity	116
4.6.3	The Class of Curves Parametrizable by Lines	118
4.7	Parametrization by Adjoint Curves	119
4.8	Symbolic Treatment of Parametrization	136
	Exercises	145
5	Algebraically Optimal Parametrization	149
5.1	Fields of Parametrization	150
5.2	Rational Points on Conics	154
5.2.1	The Parabolic Case	155
5.2.2	The Hyperbolic and the Elliptic Case	156
5.2.3	Solving the Legendre Equation	157
5.3	Optimal Parametrization of Rational Curves	169
	Exercises	185
6	Rational Reparametrization	187
6.1	Making a Parametrization Proper	188
6.1.1	Lüroth's Theorem and Proper Reparametrizations	188
6.1.2	Proper Reparametrization Algorithm	190
6.2	Making a Parametrization Polynomial	194
6.3	Making a Parametrization Normal	200
	Exercises	207
7	Real Curves	209
7.1	Parametrization	209
7.2	Reparametrization	217
7.2.1	Analytic Polynomial and Analytic Rational Functions ..	217
7.2.2	Real Reparametrization	220
7.3	Normal Parametrization	226
	Exercises	235
A	The System CASA	239

B Algebraic Preliminaries 247

 B.1 Basic Ring and Field Theory 247

 B.2 Polynomials and Power Series 250

 B.3 Polynomial Ideals and Elimination Theory 253

 B.3.1 Gröbner Bases 253

 B.3.2 Resultants 254

 B.4 Algebraic Sets 256

References 257

Index 265

Table of Algorithms 269