Lecture Notes in Mathematics 1915

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Laplacian Eigenvectors of Graphs

Perron-Frobenius and Faber-Krahn Type Theorems

Springer

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Library of Congress Control Number: 2007929852

Mathematics Subject Classification (2000): 05C50, 05C05, 05C35, 05C75, 15A18, 05C22

ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692
ISBN 978-3-540-73509-0 Springer Berlin Heidelberg New York
DOI 10.1007/978-3-540-73510-6

Springer is a part of Springer Science+Business Media
springer.com

Typesetting by the authors and SPi using a Springer LaTeX macro package
Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 12087976 41/SPi 5 4 3 2 1 0

Preface

Eigenvectors of graph Laplacians are a rather esoteric topic for a book. In fact, we are not aware of even a single review or survey article dedicated to this topic. We have, however, two excuses: (1) There are fascinating subtle differences between the properties of solutions of Schrödinger equations on manifolds on the one hand, and their discrete analogs on graphs. (2) "Geometric" properties of (cost) functions defined on the vertex sets of graphs are of practical interest for heuristic optimization algorithms. Lov Grover's observation that the cost functions of quite a few of the well-studied combinatorial optimization problems are eigenvector of associated graph Laplacians prompted us to investigate such eigenvectors more systematically.

The book in essence covers two topics: Nodal domain theorems which give bounds on the number of connected subgraphs on which an eigenvector does not change sign, and Faber-Krahn-type inequalities which are concerned with the shape of domains (i.e., graphs in our setting) with fixed volume that minimize the first Dirichlet eigenvalue. The connecting theme between these two topics is focus on local and global properties of the eigenvectors (rather than eigenvalues) and convenience of the Rayleigh quotient in the proofs.

The mindful reader will find that more often than not a simple star graph already provides a counterexample for "obvious" conjectures. In fact, we used the Petersen graph just because it seems against tradition to write about graph theory without using the Petersen graph as a counterexample at least once. The simplicity of the counterexamples highlights how little we know about the universe of graph Laplacian eigenvectors (and fitness landscapes in general), and how misguided an intuition trained on well-behaved manifolds can be in this realm: even small moves frequently causes a broken nose caused by some unexpected wall.

The history of this monograph goes back more than a decade and has its roots in the interdisciplinary research environment at the Department of Theoretical Chemistry at the University of Vienna, Austria. A collaboration with Brian Davies during his stay at the Erwin Schrödinger Institute in Vienna in 1995 stimulated our interest in Laplacian eigenvectors and eventually

resulted in a research grant from the Austrian *Fonds zur Förderung der Wissenschaftlichen Forschung* (project no. 14094-MAT) to investigate this topic in a more systematic way. Over the years, many colleagues contributed through helpful discussions, among them Wim Hordijk, Jürgen Jost, Bojan Mohar, Tomaž Pisanski, Dan Rockmore, and Gerhard Wöginger. We also thank the Max Planck Institute for Mathematics in the Sciences in Leipzig for their hospitality and for providing a fruitful scientific working for one of us (TB).

Leipzig,
Wien,
May 2006

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