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Stratified Lie Groups and Potential Theory for their Sub-Laplacians

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To Professor Bruno Pini

and to our Families

Preface

With this book we aim to present an introduction to the stratified Lie groups and to their Lie algebras of the left-invariant vector fields, starting from basic and elementary facts from linear algebra and differential calculus for functions of several real variables. The second aim of this book is to perform a potential theory analysis of the *sub-Laplacian* operators

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2,$$

where the X_j's are vector fields, i.e. linear first order partial differential operators, generating the Lie algebra of a stratified Lie group.

In recent years, these operators have received considerable attention in literature, mainly due to their basic rôle in the theory of subelliptic second order partial differential equations with semidefinite characteristic form.

1. Some Historical Overviews

General second order partial differential equations with non-negative and degenerate characteristic form have appeared in literature since the early 1900s. They were first studied by M. Picone, who called them *elliptic-parabolic equations* and proved the celebrated weak maximum principle for their solutions [Pic13,Pic27].

The interest in this type of equations in application fields was originally found by A.D. Fokker, M. Planck and A.N. Kolmogorov. They discovered that partial differential equations with non-negative characteristic form arise in the mathematical modeling of theoretical physics and of diffusion processes [Fok14,Pla17,Kol34].

Since then, over the past half-century, this type of equations appeared in many other different research fields, both theoretical and applied, including geometric theory of several complex variables, Cauchy–Riemann geometry, partial differential equations, calculus of variations, quasiconformal mappings, minimal surfaces and convexity in sub-Riemannian settings, Brownian motion, kinetic theory of gases, mathematical models in finance and in human vision. We report a short list of references for these topics at the end of this preface.

A first systematic study of boundary value problems for wide classes of elliptic-parabolic operators was performed by G. Fichera. In 1956 [Fic56a,Fic56b], he proved existence theorems of *weak solutions* of the "Dirichlet problem" and found the right subset of the boundary on which the data have to be prescribed.

Some years later, several existence and regularity results for elliptic-parabolic operators were proved by O.A. Oleĭnik and E.V. Radkevič and by J.J. Kohn and L. Nirenberg (see the monograph [OR73] for a presentation and a wide survey on this subject). The methods used by these authors required particular assumptions on the *Fichera boundary set* and led to regularity results strongly depending on the regularity of the boundary data.

1.1. L. Hörmander's Theorem

The investigations of the *local* regularity properties of the solutions to elliptic-parabolic equations, that is, regularity properties *only depending on the given operator*, have produced more interesting results. The most beautiful ones have been obtained for elliptic-parabolic equations with underlying algebraic-geometric structures of sub-Riemannian type. The milestone of these research field is a celebrated theorem of L. Hörmander proved in 1967.

Theorem 1 (L. Hörmander, [Hor67]). *Let $X_1, \dots, X_m$ and Y be smooth vectors fields, i.e. linear first order partial differential operators with smooth coefficients in the open set $\Omega \subseteq \mathbb{R}^N$. Suppose*

$$\operatorname{rank}\big(\mathrm{Lie}\{X_1, \dots, X_m, Y\}(x)\big) = N \quad \forall\, x \in \Omega. \tag{P.1}$$

Then the operator

$$L = \sum_{j=1}^{m} X_j^2 + Y \tag{P.2}$$

is hypoelliptic in Ω, i.e. every distributional solution to $Lu = f$ is of class C^∞ whenever f is of class C^∞.

Condition (P.1) simply means that at any point of Ω one can find N *linearly independent* differential operators among $X_1, \dots, X_m, Y$ and all their commutators (the Lie algebra generated by $\{X_1, \dots, X_m, Y\}$).

Hörmander's work opened up a research field, the most remarkable contributions to which have been given by G.B. Folland, L.P. Rothschild and E.M. Stein. They developed and applied to (P.2) the singular integral theory in nilpotent Lie groups.[1]

[1] The application of this theory also occurs in the developments started from the works by J.J. Kohn on the $\bar{\partial}$-Neumann problem and the $\bar{\partial}_b$ complex.

By using these techniques, in 1975, G.B. Folland accomplished a functional analytic study of sub-Laplacians on stratified Lie groups [Fol75]. One year later L.P. Rothschild and E.M. Stein proved their celebrated *lifting theorem* (see [RS76]), enlightening the basic rôle played by the sub-Laplacians in the theory of second order partial differential equations which are sum of squares of vector fields. In force of this theorem, indeed, we can roughly say that:

> *Every operator* $L = \sum_{j=1}^{m} X_j^2$ *satisfying the Hörmander rank condition* (P.1) *can be* lifted *to an operator* $\widehat{L}$ *"as close as we want" to a sub-Laplacian.*

1.2. The Rank Condition

The geometrical meaning of the rank condition (P.1) is clarified by the C. Carathéodory, W.L. Chow and P.K. Rashevsky theorem:

If (P.1) *is satisfied, then given two points* $x, y \in \Omega$*, sufficiently close, there exists a piecewise smooth curve, contained in* Ω *and connecting* x *and* y*, which is the sum of integral trajectories of the vector fields* $\pm X_1, \dots, \pm X_m, \pm Y$.

The appearance of (P.1) in Hörmander's theorem seems to be suggested by some deep properties of the Kolomogorov operators (see also the Introduction in [Hor67]), which we now aim to discuss.

In studying diffusion phenomena from a probabilistic point of view, A.N. Kolmogorov showed that the probability density of a system with $2n$ degrees of freedom satisfies an equation with non-negative characteristic form

$$Ku = 0 \quad \text{in } \mathbb{R}^{2n} \times \mathbb{R},$$

where $\mathbb{R}^{2n}$ is the phase-space of the system. A prototype for K is the following operator

$$K = \sum_{j=1}^{n} \partial_{x_j}^2 + \sum_{j=1}^{n} x_j \, \partial_{y_j} - \partial_t, \tag{P.3}$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ denote the velocity and the position vectors of the system, respectively. The operator K is "very degenerate": its second order part only contains derivatives with respect to the variables $x_1, \dots, x_n$. Nevertheless, as Kolmogorov showed, it has a *fundamental solution* Γ which is smooth out of its pole. This implies that K is hypoelliptic, that is, every distributional solution to $Ku = f$ is of class C^∞ whenever f is of class C^∞. The explicit expression of Γ is given by

$$\Gamma(z, t; \zeta, \tau) = \gamma\big(\zeta - E(t - \tau)z, t - \tau\big), \quad z = (x, y), \ \zeta = (\xi, \eta), \tag{P.4}$$

where $\gamma(z, t) = 0$ if $t \le 0$, and

$$\gamma(z, t) = \frac{(4\pi)^n}{\sqrt{\det C(t)}} \exp\Big(-\frac{1}{4}\big\langle C^{-1} z, z\big\rangle\Big) \quad \text{if } t > 0. \tag{P.5}$$

Here, $\langle\cdot,\cdot\rangle$ stands for the usual inner product in $\mathbb{R}^{2n}$; $E(t)$ and $C(t)$, respectively, denote the $2n \times 2n$ matrices

$$E(t) = \exp\left(-t\begin{pmatrix} 0 & 0 \\ \mathbb{I}_n & 0 \end{pmatrix}\right), \quad C(t) = \int_0^t E(s)A\,E(s)^T\,\mathrm{d}s.$$

Moreover, $\mathbb{I}_n$ denotes the identity matrix of order n and $A = \left(\begin{smallmatrix} \mathbb{I}_n & 0 \\ 0 & 0 \end{smallmatrix}\right)$. We explicitly remark that

$$C(t) > 0 \quad \text{for every } t > 0. \tag{P.6}$$

This condition makes expression (P.5) meaningful and can be restated in geometrical–differential terms. Indeed, denoting

$$X_j = \partial_{x_j} \quad \text{and} \quad Y = \textstyle\sum_{k=1}^n x_k\,\partial_{y_k} - \partial_t,$$

it can be proved that (P.6) *is equivalent to the following rank condition*:

$$\operatorname{rank}\big(\operatorname{Lie}\{X_1,\dots,X_n,Y\}(z,t)\big) = 2n+1 \quad \forall\,(z,t) \in \mathbb{R}^{2n+1}. \tag{P.7}$$

It is also worthwhile to note that the Kolmogorov operator K can be written as

$$K = \sum_{j=1}^n X_j^2 + Y. \tag{P.8}$$

1.3. The Left Translation and Dilation Invariance

The structure (P.4) of Kolmogorov's fundamental solution suggests the relevance that a *Lie group theoretical approach* has in the analysis of Hörmander operators. Indeed, from the explicit expression of Γ one realizes that

$$\Gamma(z,t;\zeta,\tau) = \gamma\big((\zeta,\tau)^{-1} \circ (z,t)\big),$$

where $\circ$ is the following composition law making $\mathbb{K} := (\mathbb{R}^{2n} \times \mathbb{R}, \circ)$ a *non-commutative Lie group*

$$(z,t) \circ (z',t') := \big(z' + E(t')\,z,\, t+t'\big),$$

i.e. more explicitly,

$$(x,y,t) \circ (x',y',t') = \big(x+x',\, y+y'+t'\,x,\, t+t'\big).$$

In $\mathbb{K}$ one has $(\zeta,\tau)^{-1} = (-E(-t)\,\xi, -\tau)$. It is easy to check that K is invariant w.r.t. the left translations on $\mathbb{K}$ and commutes with the following *dilations*:

$$d_\lambda(z,t) := (\lambda x, \lambda^3 y, \lambda^2 t), \quad \lambda > 0.$$

For every $\lambda > 0$, d_λ is an automorphism of $\mathbb{K}$, so that $(\mathbb{R}^{2n} \times \mathbb{R}, \circ, d_\lambda)$ is a *homogeneous Lie group*. It can be seen that its *Lie algebra* is the one generated by the vector fields $X_j = \partial_{x_j}$ and $Y = \sum_{k=1}^n x_k\,\partial_{y_k} - \partial_t$ appearing in (P.8).

1.4. The Elliptic Counterpart: Stratified Groups and Sub-Laplacians

For a proper comprehension and appreciation of this type of "parabolic"-type operators such as the above Kolmogorov operator K, it is crucial to possess a deep knowledge of their "elliptic" counterpart. This seems unavoidable, also bearing in mind that the underlying algebraic–geometric structures of these two different classes of operators are almost identical. Let us go back again, for a moment, to the Kolmogorov operator (P.3). If in that operator we square the term $Y = \sum_{j=1}^{n} x_j\, \partial_{x_{j+n}} - \partial_t$, we obtain the following "sum of square"-operator (which we may refer to as the "elliptic counterpart" of K):

$$\mathcal{L} := \sum_{j=1}^{n} \partial_{x_j}^2 + \left(\sum_{j=1}^{n} x_j\, \partial_{x_{j+n}} - \partial_t \right)^2. \tag{P.9}$$

The characteristic form of $\mathcal{L}$ is a non-negative quadratic form with non-trivial kernel. Then $\mathcal{L}$ has to be considered as a *degenerate elliptic* operator. However, it is *hypoelliptic*: the Hörmander rank condition (P.7) does not distinguish between $\mathcal{L}$ and K! Moreover, $\mathcal{L}$ is left-invariant on $(\mathbb{R}^{2n} \times \mathbb{R}, \circ)$ (as we already know, so are the ∂_{x_j}'s and Y) but, this time, it commutes with the dilations

$$\delta_\lambda(z,t) = (\lambda x, \lambda^2 y, \lambda^2 t), \quad \lambda > 0.$$

Also these dilations are automorphisms of $(\mathbb{R}^{2n} \times \mathbb{R}, \circ)$, and $\mathbb{G} := (\mathbb{R}^{2n} \times \mathbb{R}, \circ, \delta_\lambda)$ becomes a *stratified Lie group* whose *generators* are the vector fields ∂_{x_j}'s and Y. Then, according to our general agreement, $\mathcal{L}$ is a *sub-Laplacian*[2] on $\mathbb{G}$.

1.5. The Heisenberg Group

In the lower-dimensional case $n = 1$, the operator (P.9) is

$$\partial_x^2 + (x\, \partial_y - \partial_t)^2, \quad (x,y,t) \in \mathbb{R}^3. \tag{P.10}$$

Up to a change and a relabeling of the variables, this can be written as follows:

$$(\partial_x^2 + 2\, y\, \partial_t)^2 + (\partial_y - 2\, x\, \partial_t)^2, \quad (x,y,t) \in \mathbb{R}^3,$$

which, in turn, is the lower-dimensional version of the celebrated sub-Laplacian on the *Heisenberg group*.

The Heisenberg group $\mathbb{H}^n$ is the stratified Lie group $(\mathbb{R}^{2n+1}, \circ)$ whose composition law is given by

$$(z,t) \circ (z',t') = \big(z + z', t + t + 2\, \mathrm{Im}\langle z, z'\rangle\big). \tag{P.11}$$

Here we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, and we use the notation

[2] All these notions will be properly introduced in Chapter 1.

$$(z, t) = (z_1, \dots, z_n, t) = (x_1, y_1, \dots, x_n, y_n, t)$$

for the points of $\mathbb{H}^n$. In (P.11), $\langle \cdot, \cdot \rangle$ stands for the usual Hermitian inner product in $\mathbb{C}^n$. The dilation

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t) \tag{P.12}$$

is an automorphism of $\mathbb{H}^n$ and the vector fields

$$X_j = \partial_{x_j} + 2\, y_j\, \partial_t, \qquad Y_j = \partial_{y_j} - 2\, x_j\, \partial_t$$

are left-invariant on $(\mathbb{H}^n, \circ)$. One readily recognizes that the following commutation relations hold:

$$[X_j, Y_j] = -4\, \partial_t \tag{P.13a}$$

and

$$[X_j, X_k] = [Y_j, Y_k] = [X_j, Y_k] = 0 \quad \forall\, j \neq k. \tag{P.13b}$$

Identity (P.13a) is the canonical commutation relation between momentum and position in quantum mechanics. From (P.13a) it follows that

$$\operatorname{rank}\big(\operatorname{Lie}\{X_1, \dots, X_n, Y_1, \dots, Y_n, \partial_t\}\big) = 2n + 1$$

at any point of $\mathbb{R}^{2n+1}$. Then, by Hörmander's Theorem 1, the sub-Laplacian on $\mathbb{H}^n$

$$\Delta_{\mathbb{H}^n} := \sum_{j=1}^{n} (X_j^2 + Y_j^2)$$

is hypoelliptic. The Heisenberg group and its Lie algebra originally arose in the mathematical formalizations of quantum mechanics (see H. Weyl [Weyl31]). Today, they appear in many research fields such as several complex variables, CR geometry, Fourier analysis and partial differential equations of subelliptic type. The Heisenberg sub-Laplacian is undoubtedly the most important prototype of the sub-Laplacians on non-commutative stratified Lie groups.

1.4. The Lifting Theorem

Obviously, a generic Hörmander operator *sum of squares of vector fields* is not, in general, a sub-Laplacian on some stratified Lie group. Just consider, as an example,

$$M = \partial_x^2 + (x\, \partial_y)^2 \quad \text{in } \mathbb{R}^2.$$

This operator satisfies the Hörmander rank condition, hence it is hypoelliptic. However, there is no Lie group structure in $\mathbb{R}^2$ making M left-invariant on it. Nevertheless, adding the new variable t, M can be *lifted* to the operator in (P.10) which is the sub-Laplacian on (a group isomorphic to) the Heisenberg group $\mathbb{H}^1$. This is the source idea of the *lifting theorem* by L.P. Rothschild and E.M. Stein, which states, roughly speaking, that any Hörmander operator sum of squares of vector fields can be approximated by a sub-Laplacian on a stratified group. This result emphasizes the major rôle played by the sub-Laplacians in the theory of second order PDE's with non-negative and degenerate characteristic form.

1.5. Stratified Groups in Sub-Riemannian Geometry

Stratified groups also appear naturally in sub-Riemannian geometry (frequently referred to as “Carnot” geometry). Roughly speaking, stratified groups play a rôle, for sub-Riemannian manifolds, analogous to that played by Euclidean vector spaces for Riemannian manifolds.

More precisely, once it has been provided a suitable notion of *tangent space* at a point of a sub-Riemannian manifold, it turns out that (at a regular point) this tangent space is naturally endowed with a structure of nilpotent Lie group with dilations, a stratified Lie group (see J. Mitchell [Mit85] and A. Bellaïche in [BR96]).

Furthermore, the analysis of a left invariant sub-Laplacian on a connected nilpotent Lie group (or more generally on a Lie group *of polynomial growth*) and the geometry at infinity of this group is described by a canonically associated dilation-invariant sub-Laplacian on a stratified Lie group. See G. Alexopoulos [Ale92,Ale02], S. Ishiwata [Ish03] and N.Th. Varopoulos [Varo00].

2. The Contents of the Book: An Overview

A glance at the contents of the book and at our approach to the subjects is in order. The book is divided into three parts, and every part is, in its turn, subdivided in several chapters plus some appendices, if necessary.

2.1. Part I

The first four chapters of Part I are devoted to an elementary and self-contained introduction to the stratified Lie groups in $\mathbb{R}^N$. Our presentation does not require a specialized knowledge neither in algebra nor in differential geometry. The approach is completely elementary, “constructive” whenever possible, abundant in examples and intended to be understandable by readers with basic backgrounds only in linear algebra and differential calculus in $\mathbb{R}^N$. Subsequently, we present the formal and abstract approach to the stratified Lie groups commonly used in literature, and we prove the equivalence of the abstract notion of *stratified group* to the “constructive” notion of *homogeneous Carnot group*. This equivalence is also provided in Part I.

A very special emphasis is given to the examples. We introduce and discuss a wide range of explicit stratified Lie groups of arbitrarily large dimension and step. Some of them have been known in specialized literature for several years, such as the Heisenberg–Kaplan groups, the filiform groups and the Métivier groups. Many others have only appeared very recently, in particular what we shall call the Kolmogorov-type groups and the Bony-type groups. Other examples are completely new, some extracted from geometric control theory.

Our long list of examples is also intended to be appreciated by readers working in geometry and analysis on Carnot groups. It provides a valuable benchmark set to test new special properties of the groups, to exhibit explicit examples and counterexamples of the "pathologies" and the special features of Carnot groups.

It is also payed a special attention to the Lie algebras of the groups by stressing their links with second order partial differential operators of Hörmander type (sum of squares of vector fields). In particular, given such an operator, we show necessary and sufficient conditions for it to be a sub-Laplacian on a suitable homogeneous stratified Lie group, and we explicitly show how to construct the related composition law. As a byproduct, this enables the reader to build up another plenty of examples of stratified groups and sub-Laplacians.

Chapter 5 of Part I is dedicated to the analysis of the *fundamental solution* for the sub-Laplacians, a central topic of Part I. Here, the mainly used analytic tools are *integration by parts* and *coarea-formulas*. We start from the *hypoellipticity* of sub-Laplacians, easy consequence[3] of the Hörmander Theorem 1.

From this "assumption" on hypoellipticity, and with the aid of the *strong maximum principle*, whose proof is postponed to the Appendix of Chapter 5, we deduce the existence of a *gauge function* d for any given sub-Laplacian $\mathcal{L}$, i.e. the existence of a positive non-constant homogeneous function d such that d^{2-Q} is $\mathcal{L}$*-harmonic* away from the origin. Here Q stands for the homogeneous dimension of the group on which the sub-Laplacian lives (*we always assume that* $Q \geq 3$).

This property is one of the most striking analogies between $\mathcal{L}$ and the classical Laplace operator. We show that this leads to suitable *mean value formulas* on the *d-balls*, extending to this new setting the well-known Gauss theorem for classical harmonic functions. We then use these formulas (which will play a crucial rôle throughout the book) to prove *Liouville-type theorems*, *Harnack-type inequalities*, and a *Sobolev–Stein embedding theorem*. Furthermore, three sections are devoted to the following topics: some remarks on the analytic-hypoellipticity of sub-Laplacians, $\mathcal{L}$-harmonic approximations and an integral representation formula for the fundamental solution.

2.2. Part II

Part II of the book contains an exhaustive potential theory for the sub-Laplacians. Basically, our only starting point is the theorem by G.B. Folland asserting the *existence* for these operators of a *homogeneous* and *smooth* fundamental solution with pole at the origin. This key result allows us to perform a complete potential theory that parallel the one of the classical Laplace operator.

The lack of explicit Poisson integral formulas forces us to follow an abstract approach to this theory. For this reason, in Chapter 6 we present some topics from abstract harmonic space theory, mainly inspired by the ones developed by H. Bauer [Bau66] and C. Costantinescu and A. Cornea [CC72]. This chapter mainly involves Perron–Wiener–Brelot method for the Dirichlet problem, harmonic minorants and majorants and balayage theory.

[3] We do not go into the proof of this theorem, for it would require techniques very far from the ones developed in the book.

Next, in Chapter 7 we show that every sub-Laplacian equips $\mathbb{R}^N$ with a structure of harmonic space satisfying the axioms of the theory presented in Chapter 6. This is accomplished by using the Harnack-type theorem proved in Chapter 5, and then by showing the existence of a basis of the topology of $\mathbb{R}^N$ formed by $\mathcal{L}$-regular sets, i.e. by sets for which the Dirichlet problem for $\mathcal{L}$ is solvable in the usual classical sense (here we follow an idea by J.-M. Bony [Bon69]).

In the subsequent chapters of Part II, we use the full strength of the abstract theory, together with the remarkable properties of the fundamental solution for $\mathcal{L}$ to deal with the arguments listed below:

a) sub-mean characterizations of the $\mathcal{L}$-subharmonic functions, and applications to the notion of convexity in Carnot groups;
b) Green functions and Riesz representation theorems for $\mathcal{L}$-subharmonic functions, with applications (among which Bôcher-type theorems);
c) maximum principles on unbounded domains;
d) $\mathcal{L}$-capacity and $\mathcal{L}$-polar sets, with applications: the Poisson–Jensen formula and the so-called *fundamental convergence theorem*;
e) $\mathcal{L}$-thinness and $\mathcal{L}$-fine topology, with applications to the Dirichlet problem (and the derivation of Wiener's criterion);
f) the links between the Hausdorff measure naturally related to the gauge d and the capacity for $\mathcal{L}$.

In writing this part of the book we were also partially inspired by some monographs on potential theory for the classical Laplace operator—in particular the beautiful books by L.L Helms [Helm69], by W.K. Hayman and P.B. Kennedy [HK76] and by D.H. Armitage and S.J. Gardiner [AG01].

2.3. Part III

In Part III, we take up further topics on the algebraic and analytic theory of Carnot groups. In particular, this part of the book provides:

a) the study of *free Lie algebras*;
b) clear and complete proofs, in several contexts, of the fundamental and remarkable *Campbell–Hausdorff formula*[4];
c) the equivalence of sub-Laplacians under diffeomorphisms;
d) the *Rothschild–Stein lifting theorem* and *Folland's lifting theorem* (for stratified or homogeneous vector fields);
e) the study of the algebraic structure of *Heisenberg–Kaplan-type groups* (also providing an explicit characterization of them) with a special emphasis to the remarkable form of their fundamental solutions, discovered by G.B. Folland and A. Kaplan (we also present the *inversion* and the *Kelvin transform* in the H-type groups of *Iwasawa type*);

[4] In Chapter 15, we collect *four* theorems for the Campbell–Hausdorff formula: one for homogeneous vector fields, two for formal power series and one for general smooth vector fields.

f) the *Carathéodory–Chow–Rashevsky connectivity theorem* (for stratified vector fields) with applications;
g) *Taylor's formula* (with Lagrange and with integral remainder) on Carnot groups.

The difficulty of finding "easy" and complete proofs of some of the above mentioned results in the existing literature is well known. By working with stratified vector fields we are able to overcome some of the lengthy steps of the proofs, while maintaining a good amount of generality in the final results.

3. How to Read this Book

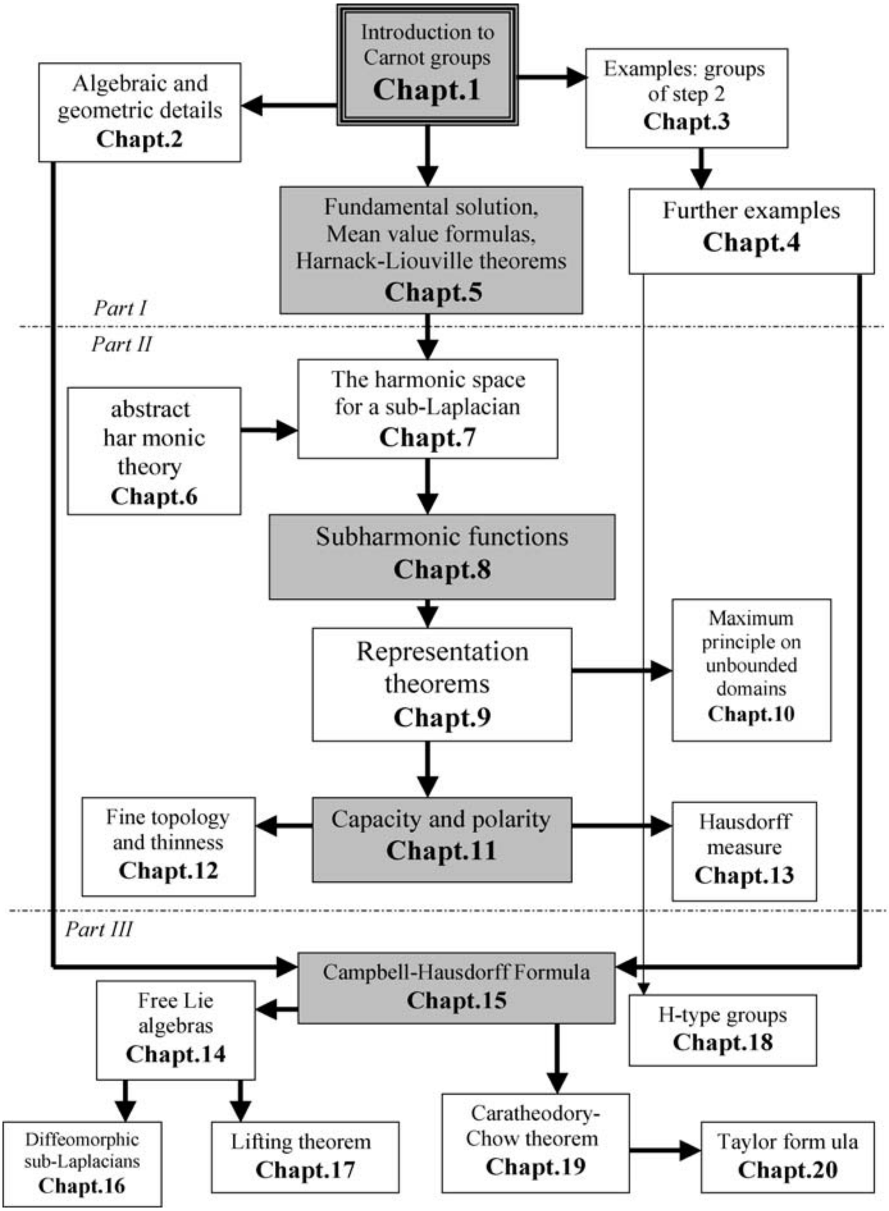
Besides Ph.D. students, the book is addressed to young and senior researchers. Indeed, one of the main efforts in presenting the material is to use an elementary approach and to reach, step by step, the level of current researches. Many parts of the book may be used for graduate courses and advanced lectures.

The first four chapters of Part I are addressed to non-specialists in Lie group and Lie algebra theory. The first two chapters can be skipped by the readers having familiarity with the basics of differential geometry and Lie group theory. The reader already acquainted with Carnot group theory can pass directly to Chapter 5. In any case, beginners and specialists in the theory of stratified groups can exploit the first four chapters as a source for examples.

Part II is the core of the monograph. The reader with some background in potential theory (and interested in the main case of sub-Laplacians) can pass directly to Chapter 7 and proceed throughout Part II, leaving Chapters 10 and 13 as a further reading.

Part III is thought of as a more specialized lecture. Nonetheless, a deep understanding of, e.g. the Campbell–Hausdorff formula or of Heisenberg-type groups are amongst the main goals of this monograph.

The book provides 21 illustrative figures, 250 exercises (each chapter has its own section of exercises) and an index of the basic notation. For the reading convenience, we furnish a synoptic diagram of the structure of the book on page XVII.



The synoptic diagram of the structure of the book.

4. Some References on Theoretical and Applied Related Topics

Here is a short list of references for related topics on analysis on stratified Lie groups and applications.[5]

Alexopoulos [Ale02], Altafini [Alt99], Bellaïche and Risler [BR96],
Birindelli, Capuzzo Dolcetta and Cutrì [BCC97], Bahri [Bah04,Bah03],
Barletta [Bar03], Barletta and Dragomir [BD04],
Barletta, Dragomir and Urakawa [BDU01],
Birindelli, Capuzzo Dolcetta and Cutrì [BCC98],
Brandolini, Rigoli and Setti [BRS98], Capogna [Cap99],
Capogna and Cowling [CC69], Capogna and Garofalo [CG98,CG03,CG06],
Capogna, Garofalo and Nhieu [CGN00,CGN02], Capuzzo Dolcetta [CD98],
Chandresekhar [Cha43], Citti [Cit98], Citti, Lanconelli and Montanari [CLM02],
Citti, Manfredini and Sarti [CMS04], Citti and Montanari [CM00],
Citti, Pascucci and Polidoro [CPP01], Citti and Sarti [CS06],
Citti and Tomassini [CT04] Cowling, De Mari, Korányi and Reimann [CDKR02],
Cowling and Reimann [CR03], Danielli, Garofalo, Nhieu and Tournier [DGNT04],
Danielli, Garofalo and Salsa [DGS03], Dragomir [Dra01],
Franchi, Gutiérrez and van Nguyen [FGvN05],
Franchi, Serapioni and Serra Cassano [FSS03a,FSS03b,FSS01],
Gamara [Gam01], Gamara and Yacoub [GY01],
Garofalo and Lanconelli [GL92], Garofalo and Tournier [GT06],
Garofalo and Vassilev [GV00], Golé and Karidi [GK95],
Gutiérrez and Lanconelli [GL03], Gutiérrez and Montanari [GM04a,GM04b],
Heinonen [Hei95b], Heinonen and Holopainen [HH97],
Heinonen and Koskela [HK98], Huisken and Klingenberg [HK99],
Jerison and Lee [JL87,JL88,JL89],
Juutinen, Lu, Manfredi and Stroffolini [JLMS07],
Korányi and Reimann [KR90,KR95],
Lanconelli [Lan03], Lanconelli, Pascucci and Polidoro [LPP02],
Lanconelli and Uguzzoni [LU00], Lu, Manfredi and Stroffolini [LMS04],
Lu and Wei [LW97], Malchiodi and Uguzzoni [MU02],
Manfredi and Stroffolini [MS02], Montanari [Mo01],
Montanari and Lascialfari [ML01], Montgomery [Mon02],
Montgomery, Shapiro and Stolin [MSS97], Monti and Morbidelli [MM05],
Monti and Rickly [MR05], Monti and Serra Cassano [MSC01],
Petitot and Tondut [PT99], Reimann [Rei01a,Rei01b],
Slodkowski and Tomassini [ST91], Stein [Ste81],
Uguzzoni [Ugu00], Varopoulos, Saloff-Coste and Coulhon [VSC92].

[5] The list is alphabetically ordered and the grouping of the references in different lines is only meant for typographical readability.

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