Lecture Notes in Economics
and Mathematical Systems

Founding Editors:
M. Beckmann
H.P. Künzi

Managing Editors:
Prof. Dr. G. Fandel
Fachbereich Wirtschaftswissenschaften
Fernuniversität Hagen
Feithstr. 140/AVZ II, 58084 Hagen, Germany

Prof. Dr. W. Trockel
Institut für Mathematische Wirtschaftsforschung (IMW)
Universität Bielefeld
Universitätsstr. 25, 33615 Bielefeld, Germany

Editorial Board:
A. Basile, A. Drexl, H. Dawid, K. Inderfurth, W. Kürsten
Generalized Convexity and Optimization

Theory and Applications
To Giovanna and Paolo
In the latter part of the twentieth century, the topic of generalizations of convex functions has attracted a sizable number of researchers, both in mathematics and in professional disciplines such as economics/management and engineering. In 1994 during the 15th International Symposium on Mathematical Programming in Ann Arbor, Michigan, I called together some colleagues to start an affiliation of researchers working in generalized convexity. The international Working Group of Generalized Convexity (WGGC) was born. Its website at www.genconv.org has been maintained by Riccardo Cambini, University of Pisa.

Riccardo’s father, Alberto Cambini, and Alberto’s long-term colleague Laura Martein in the Faculty of Economics, University of Pisa, are the co-authors of this volume. My own contact with generalized convexity in Italy dates back to my first visit to their department in 1980, at a time when the first international conference on generalized convexity was in preparation. Thirty years later it is now referred to as GC1, an NATO Summer School in Vancouver, Canada. Currently WGGC is preparing GC9 which is to take place in Kaohsiung, Taiwan. As founding chair and also current chair of WGGC, I am delighted to see the continued interest in generalized convexity of functions, augmented by the topic of generalized monotonicity of maps.

Eight international conferences have taken place in this research area, in North America (2), Europe (5) and Asia (1). We thought it was now time to return to Asia since our membership has shifted towards Asia.

As an applied mathematician I have taught mostly in management schools. However, I am currently in the process of joining an applied mathematics department. One of the first texts I will try out with my mathematics students is this volume of my long-term friends from Pisa. I recommend this volume to anyone who is trying to teach generalized convexity/generalized monotonicity in an applied mathematics department or in a professional school. The volume is suitable as a text for both. It contains proofs and exercises. It also provides sufficient references for those who want to dig deeper as graduate students and as researchers. With dedication and much love the authors have written a
book that is useful for anyone with a limited background in basic mathematics. At the same time, it also leads to more advanced mathematics.

The classical concepts of generalized convexity are introduced in Chaps. 2 and 3 with separate sections on non-differentiable and differentiable functions. This has not been done in earlier presentations. Chapter 4 deals with the relationship of optimality conditions and generalized convexity. One of the reasons for a study of generalized convexity is that convexity usually is just a convenient sufficient condition. In fact most of the time it is not necessary. And it is a rather rigid assumption, often not satisfied in real-world applications. That is the reason why economists have replaced it by weaker assumptions in more contemporary studies. In fact, some of the progress in this research area is due to the work of economists. I am glad that the new book emphasizes economic applications.

In Chap. 5 the transition from generalized convexity to generalized monotonicity occurs. Historically, this happened only around 1990 when I was working with the late Stepan Karamardian after joining the University of California at Riverside. He was a former PhD student of George Dantzig at the University of California at Berkeley. We collaborated on the last two papers he published, both on generalized monotonicity. (a new research area) We had opened up together.

In 2005 Nicolas Hadjisavvas, Sandor Komlosi and I completed the first Handbook of Generalized Convexity and Generalized Monotonicity with contributions from many leading experts in the field, including Alberto Cambini and Laura Martein, a proven team of co-authors who in their unique colorful way have left an imprint in the field. The new book is further evidence of their style.

Chapters 6 and 7 are devoted to specialized results for quadratic functions and fractional functions. With this the authors follow the outline of the first monograph in this research area, Generalized Concavity by Mordecai Avriel, Walter E. Diewert, Siegfried Schaible and Israel Zang in 1988. Chapter 8 contains algorithmic material on solving generalized convex fractional programs. It defeats the objection sometimes raised that the area of generalized convexity lacks algorithmic contributions. It is true that there could be more results in this important direction on a topic which by nature is theoretical. Perhaps the presentation in Chap. 8 will motivate others to take up the challenge to derive more results with a computational emphasis.

Today Generalized Concavity (1988) is available to us as the first volume on the topic, together with the comprehensive Handbook of Generalized Convexity and Generalized Monotonicity (2005), an edited volume of 672 pages, written by 16 different researchers including Alberto Cambini and Laura Martein. In addition, the published proceedings of GC1–GC8 are available from reputable publishing houses. The proceedings of GC9 will appear partially in the prestigious Taiwanese Journal of Mathematics.

As somebody who has participated in all the conferences, GC1–GC8, and who is co-organizing GC9 together with Jen-Chih Yao, Kaohsiung and
who has been involved in most publications mentioned before, I congratulate
the authors for having produced such a fine volume in this growing area of
research. Like me they stumbled into it when no monographs on the topic
were available. I can see the usefulness of the book for teaching and research
for generations to come. Its technical level makes it suitable for undergraduate
and graduate students. The level is pitched wisely. The book is more accessible
than the Handbook as it assumes less background knowledge about the topic.
This is not surprising as the purpose of the Handbook is different. The new
book can serve as an up-to-date link to the Handbook. It also saves the reader
from going through the earlier proceedings with more dated results.

As someone who, like the authors, has not departed from the area of
generalized convexity in his career, I can highly recommend this excellent new
volume in our community of researchers. WGGC has been the background for
most recent publications in our field of study. It is the excitement of working
in teams which has been promoted by WGGC. A sense of community very
common in Italy is the background of this new volume. It made me happy
when I reviewed the manuscript first. I hope that many readers will come to
the same conclusion. My thanks and congratulations go to the authors for a
job well done.

I want to thank the authors for having taken the time to write *Generalized
Convexity and Optimization with Economic Applications* and for their diligent
effort to produce an up-to-date text and wish the book much success among
our growing community of researchers.

Riverside, California,  
June 2008

*Siegfried Schaible*  
Chair of WGGC
## Contents

1 Convex Functions .............................................. 1
   1.1 Introduction ............................................ 1
   1.2 Convex Sets ............................................ 1
      1.2.1 Topological Properties of Convex Sets ............. 3
      1.2.2 Relative Interior of Convex Sets ..................... 4
      1.2.3 Extreme Points and Extreme Directions .............. 4
      1.2.4 Supporting Hyperplanes and Separation Theorems ..... 6
      1.2.5 Convex Cones and Polarity .......................... 7
   1.3 Convex Functions ....................................... 10
      1.3.1 Algebraic Structure of the Convex Functions.......... 13
      1.3.2 Composite Function ............................... 13
      1.3.3 Differentiable and Twice Differentiable Convex
           Functions ........................................ 14
   1.4 Convexity and Homogeneity .............................. 16
   1.5 Minima of Convex Functions .............................. 17
   1.6 Exercises ............................................... 18
   1.7 References .............................................. 21

2 Non-Differentiable Generalized Convex Functions ............. 23
   2.1 Introduction ............................................ 23
   2.2 Quasiconvexity and Strict Quasiconvexity ................. 23
   2.3 Semistrict Quasiconvexity ............................... 30
   2.4 Generalized Convexity of Some Homogeneous Functions ..... 34
      2.4.1 The Cobb–Douglas Function .......................... 34
      2.4.2 The Constant Elasticity of Substitution (C.E.S.)
           Function ........................................ 35
      2.4.3 The Leontief Production Function ........................ 35
      2.4.4 A Generalized Cobb–Douglas Function ................. 35
   2.5 Generalized Quasiconvex Functions in One Variable ........ 36
   2.6 Exercises ............................................... 38
   2.7 References .............................................. 40
3 Differentiable Generalized Convex Functions .......................... 41
   3.1 Introduction ..................................................... 41
   3.2 Differentiable Quasiconvex and Pseudoconvex Functions .......... 41
       3.2.1 Differentiable Quasiconvex Functions ................. 41
       3.2.2 Pseudoconvex Functions ............................. 43
       3.2.3 Relationships .................................... 48
   3.3 Quasilinearity and Pseudolinearity ............................... 50
       3.3.1 Quasilinearity and Semistrict Quasilinearity ......... 50
       3.3.2 Pseudolinearity .................................... 53
   3.4 Twice Differentiable Generalized Convex Functions ............. 57
       3.4.1 Quasiconvex Functions .............................. 57
       3.4.2 Pseudoconvex Functions ............................ 60
       3.4.3 Characterizations in Terms of the Bordered Hessian ... 61
   3.5 Generalized Convexity at a Point ................................ 63
   3.6 Exercises ...................................................... 68
   3.7 References .................................................... 71

4 Optimality and Generalized Convexity ..................................... 73
   4.1 Introduction .................................................... 73
   4.2 Necessary Optimality Conditions Via Separation Theorems ...... 73
   4.3 Generalized Convexity and Constraint Qualifications .......... 78
   4.4 Sufficiency of the Karush–Kuhn–Tucker Conditions ............ 81
   4.5 Local-Global Property ........................................ 82
   4.6 Maxima and Generalized Convexity ................................ 84
   4.7 Minima, Maxima and Pseudolinearity ............................ 86
   4.8 Economic Applications .......................................... 87
       4.8.1 The Utility Maximization Problem ...................... 89
       4.8.2 The Expenditure Minimization Problem ............... 91
       4.8.3 The Profit Maximization Problem and the Cost Minimization Problem ................. 91
   4.9 Invex Functions .................................................. 93
   4.10 Exercises ....................................................... 95
   4.11 References ..................................................... 97

5 Generalized Convexity and Generalized Monotonicity ................. 99
   5.1 Introduction .................................................... 99
   5.2 Concepts of Generalized Monotonicity .......................... 99
       5.2.1 Differentiable Generalized Monotone Maps ............ 103
   5.3 Generalized Monotonicity of Maps of One Variable ............ 103
   5.4 Generalized Monotonicity of Affine Maps ..................... 105
   5.5 Relationships Between Generalized Monotonicity and Generalized Convexity ................. 108
   5.6 The Generalized Charnes–Cooper Transformation .............. 110
   5.7 References ..................................................... 112
6 Generalized Convexity of Quadratic Functions 
6.1 Introduction .................................................. 115
6.2 Preliminary Results ............................................. 115
   6.2.1 Some Properties of a Quadratic form Associated with 
       a Symmetric Matrix Having One Simple Negative 
       Eigenvalue ........................................... 116
6.3 Quadratic Functions ............................................ 119
6.4 Quadratic Functions of Non-negative Variables .................. 126
6.5 Pseudoconvexity on a Closed Set ................................ 128
   6.5.1 Pseudoconvexity on the Non-negative Orthant ............ 130
   6.5.2 Generalized Convexity of a Quadratic form on $\mathbb{R}^2_+$ ... 131
6.6 A Special Case ............................................. 132
6.7 Exercises .................................................. 134
6.8 References ............................................... 135

7 Generalized Convexity of Some Classes of Fractional 
   Functions .................................................. 137
7.1 Introduction ............................................ 137
7.2 The Ratio of a Quadratic and an Affine Function ............ 137
7.3 The Sum of a Linear and a Linear Fractional Function ....... 141
7.4 Pseudoconvexity and the Charnes–Cooper Variable 
       Transformation ........................................... 147
7.5 Sum of Two Linear Fractional Functions ....................... 149
7.6 Exercises .................................................. 154
7.7 References ............................................... 157

8 Sequential Methods for Generalized Convex Fractional 
   Programs .................................................. 159
8.1 Introduction .................................................. 159
8.2 The Linear Fractional Problem ................................ 160
   8.2.1 Isbell–Marlow’s Algorithm ................................ 162
   8.2.2 Charnes–Cooper’s Algorithm ............................ 164
   8.2.3 Martos’ Algorithm ...................................... 166
   8.2.4 Cambini–Martein’s Algorithm ............................ 168
   8.2.5 The Case of an Unbounded Feasible Region .......... 172
8.3 A Generalized Linear Fractional Problem .................. 175
   8.3.1 Sequential Methods .................................... 176
   8.3.2 The Sum of Two Linear Fractional Functions ........ 182
8.4 Generalized Linear Multiplicative Programs ................. 183
   8.4.1 The Sum of a Linear Function and the Product 
       of Two Affine Functions .................................. 183
   8.4.2 The Product Between an Affine Function 
       and the Power of an Affine Function ............... 185
8.5 The Optimal Level Solutions Method .......................... 189
8.6 References ............................................... 193
XIV Contents

9 Solutions .................................................. 195

References ..................................................... 213

A Mathematical Review ........................................ 229
  A.1 Sets ......................................................... 229
  A.2 The Euclidean Space $\mathbb{R}^n$ ..................... 230
  A.3 Topological Concepts .................................. 234
  A.4 Functions ................................................. 235

B Concave and Generalized Concave Functions .......... 241

Index .......................................................... 247