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Zdenek Martinec

Boundary-Value Problems for Gravimetric Determination of a Precise Geoid

With 51 Figures and 3 Tables

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Author

Zdeněk Martinec
Department of Geophysics
Faculty of Mathematics and Physics
Charles University
V Holešovičkách 2, 180 00 Prague 8, Czech Republic
e-mail. zdenek@hervam.troja.mff.cuni.cz

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Contents

List of symbols

a_{1m}	Hörmander's constant
A^B	gravitational attraction of spherical Bouguer shell
A^c	gravitational attraction of compensed/condensed masses
A^t	gravitational attraction of topographical masses
δA	direct topographical effect on gravity
b_0	minor semi-axis of the reference ellipsoid best-fitting the geoid
D	depth of compensation
e	first eccentricity
e_0	first eccentricity of reference ellipsoid
E	linear eccentricity of set of confocal ellipsoidal coordinate surfaces u =const.
F	free-air reduction
F	hypergeometric function
g	gravity of the Earth
δg	gravity disturbance
Δg^F	free-air gravity anomaly
Δg^F_ℓ	low-frequency part of Δg^F
$\Delta g^{F,\ell}$	high-frequency part of Δg^F
G	Newton's gravitational constant
h	ellipsoidal height
H	topographical height
j_{ref}	cut-off degree of reference potential
j_{max}	cut-off degree of anomalous potential
K	spherical Poisson's kernel
K^{ell}	ellipsoidal Poisson's kernel
K^ℓ	spheroidal Poisson's kernel
ℓ	distance between points (r, Ω) and (R, Ω')
ℓ_0	distance between points (R, Ω) and (R, Ω')
L	distance between points (r, Ω) and (r', Ω')
L_2	space of square-integrable functions on sphere
M	mass of the Earth
M^c	mass of compensation/condensation layer
M^g	masses below the geoid
M^t	topographical masses
N	geoidal height

q_j	truncation coefficients of Poisson's kernel
q_j^ℓ	high-degree part of q_j
P	point on the Earth's surface
P_g	point on the geoid
P_j	Legendre polynomial of degree j
P_{jm}	Legendre functions of the 1st kind
Q	point on reference ellipsoid
Q_{jm}	Legendre functions of the 2nd kind
r	radial distance
r_g	radius of the geoid
R	radius of mean sphere best-fitting the geoid
R_{jk}	Paul's coefficients
S	spherical Stokes's function
S^{elco}	spherical-ellipsoidal Stokes's function
S^{ell}	ellipsoidal Stokes's function
δS	secondary indirect topographical effect on gravity
T	anomalous gravitational potential harmonic outside the Earth
T_ℓ	low-degree reference part of T
T^ℓ	high-degree part of T
T_{jm}	spherical harmonic coefficients of T
T^h	anomalous gravitational potential harmonic outside the geoid
$T^{h,\ell}$	high-degree part of T^h
u	ellipsoidal coordinate
U	normal gravity potential
V	gravitational potential of the Earth
V^B	gravitational potential of spherical Bouguer shell
V^c	gravitational potential of compensated/condensed masses
V^g	gravitational potential of masses below the geoid
V^R	terrain roughness term
V^t	gravitational potential of topographical masses
$V_{jm}^{t,e}$	spherical harmonic coefficients of V^t in the space external to topographical masses
$V_{jm}^{t,i}$	spherical harmonic coefficients of V^t in the space internal to topographical masses
V^ϱ	gravitational potential of compensated masses
V^σ	gravitational potential of condensed masses
V^ω	centrifugal potential
δV	residual topographical potential
δV_ℓ	low-degree part of δV
δV^ℓ	high-degree part of δV
δw	gravity potential disturbance
W	gravity potential of the Earth
W_0	gauge value of gravity potential on the geoid
x, y, z	Cartesian coordinates

Y_{jm}	surface spherical harmonic of degree j and order m
α	azimuth
β	reduced co-latitude
γ	normal/reference gravity
Γ	gamma function
δ_{ij}	Kronecker symbol
ϵ_h	ellipsoidal correction
ϵ_γ	ellipsoidal correction
ζ	Riemann zeta function
ϑ	co-latitude
κ	condition number
λ	longitude
λ_i	eigenvalue
ϱ	density of topographical masses
$\overline{\varrho}$	density ϱ averaged along topographical column
ϱ_c	density of compensated masses
ϱ_0	mean density of topographical masses
$\Delta\varrho_{Moho}$	density contrast at the Moho
σ	density of condensed masses
χ	angular distance in ellipsoidal coordinates
ψ	angular distance in spherical coordinates
ω	angular velocity of the Earth's rotation
Ω	pair of angular spherical coordinates
$\overline{\Omega}$	pair of angular ellipsoidal coordinates
Ω_0	full solid angle
Ω_{ψ_0}	spherical cap of radius ψ_0

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Introduction

Whether the geoid can be determined to a 'sufficient' accuracy has been discussed in geodetic circles for many years. The main objection, by those who do not think it can, has always been that the mass density distribution within the Earth will never be known accurately enough to allow us to compute the geoid to a reasonable level of accuracy. This was the main reason that Molodenskij's quasi-geoid and the theory of its determination were preferred for several decades. With the arrival of GPS and its capability to measure ellipsoidal height differences fairly accurate, the discussion on this subject has been renewed. In recent years, different groups have been trying to compute either an accurate geoid or an accurate quasi-geoid (Sjöberg, 1994; Sideris, 1994; Denker et al., 1994). The final goal of these efforts is the determination of a geoid/quasi-geoid with an error of one centimetre or less (Hipkin, 1994).

The geoid over continental areas has recently been determined by two techniques. First, combining GPS positioning with the orthometric heights results in the 'geometrical' geoid whose undulations with respect to the level ellipsoid are given as the differences of the ellipsoidal (GPS-determined) heights minus the orthometric heights (Mainville et al., 1995; Sideris and She, 1995). On the other hand, surface gravity observations supplemented by geodetic levelling can be used for constructing the 'gravimetric' geoid (Vaníček and Kleusberg, 1987; Stewart and Hipkin, 1990; Milbert, 1990; Featherstone, 1992; Forsberg and Sideris, 1993). As a matter of fact, these techniques are by no means independent, since both make a use of a density hypothesis within the Earth. Only the ellipsoidal height differences resulting from GPS positioning on one hand and gravimetric data on the other hand are independently determined. So, we face the fact that the geoid can be determined in two ways, both depending on mass density distribution within the Earth. Hence, there is a hope for a near future that this overdeterminacy will lead to an improvement of density distribution modelling in the uppermost part of the Earth. In order to reach this goal, the geoid should be determined with a 1 dm accuracy (or better) since a change of the density model in the Earth's crust from a commonly accepted constant 2.67 g/cm^3 to a more realistic 3-D density distribution changes the geoid by a few decimetres only (Martinec, 1993).

A magic accuracy of 1 dm in the determination of gravimetric geoid - not yet realizable in a mountainous terrain - requires not only highly accurate surface gravity observations but also accurate theories and corresponding numerical codes for geoid height computations. The last requirement has not been resolved satisfactory yet since existing theories for geoid computations still contain some assumptions which do not allow one to reach the desired accuracy of 1 dm.

The traditional formulation of the problem for geoid height determination over a continental area with non-zero terrain elevations comes out from the fact that heights of the terrain above the geoid and the modulus of the surface gravity are known with a certain accuracy. (This problem can simply be reformulated in the case that the surface gravity potential is given instead of the terrain heights above the geoid.) The effort is to employ Stokes's integral (Stokes, 1849) for determining the gravity potential of the geoid. To be able to use this integral, three requirements must be satisfied: there are no masses outside the geoid so that the anomalous gravitational potential is harmonic outside the geoid, the gravity observations are referred to the geoid, and the geoid is considered as a sphere in a boundary condition for the anomalous gravitational potential. A remove-compute-restore technique is one possibility which, however, satisfies these requirements only approximately:

- Linearize the boundary condition for modulus of gravity on the Earth's surface with respect to a reference gravity potential;
- Linearize boundary condition for gravity potential on the geoid with respect to a level ellipsoid;
- Replace the gravitational effect of topographical masses on gravity at the Earth's surface by that of an auxiliary mass layer/body located on/below the geoid;
- Continue harmonically the linearized boundary functional from the Earth's surface to the geoid;
- Formulate and solve classical Stokes's boundary-value problem on a spherical geoid;
- Restore the gravitational effect of topographical masses.

Unfortunately, a number of theoretical and practical problems appear when this traditional technique for gravimetric geoid determination is used to compute a 'decimetre geoid'. In this thesis, we will attempt to sort out some of these problems and show the ways how to overcome them.

In Chapter 1 we formulate the boundary-value problem for gravimetric determination of the geoid. The boundary condition constraining the solution has not a usual form, because it contains the unknown anomalous potential referred to both the Earth's surface and the geoid coupled by the topographical height. To emphasize the 'two-boundary' character and adopting the terminology introduced by Sansò (1995), this boundary-value problem is called the Stokes two-boundary-value problem. We analyse numerically the solvability of this problem, in particular, we focus on treating the existence of a short-wavelength part of solution. Moreover, we discuss various approximations frequently used in geodesy converting the 'two-boundary' condition to a 'one-boundary' condition only, referred either to the Earth's surface or to the geoid.

In order to ensure the uniqueness of a solution to the Stokes two-boundary-value problem, the first-degree harmonics of an anomalous potential we are looking for must be excluded apriorily from the solution. However, due to the compensation of topographical masses, the first-degree harmonics of the anomalous potential do not, in general, vanish. In Chapter 2 we show the way how to determine these 'forbidden' harmonics.

The effect of topographical masses on geoid height computation is discussed in Chapters 3 to 6. Adopting a spherical approximation of the geoid, we first derive new formulae for the direct and indirect topographical effects. Our expressions are formulated such that the influence of lateral variations of topographical density can be taken into account. As a by-product of our investigations, we proceed to remove the weak singularity of the Newton integral for topographical effects by introducing spherical 'Bouguer' shell. We study the influence of different approximations used in geodesy to approximate the geoid, particularly, we focus on a planar model of the geoid since this may considerably bias results by systematic errors.

The strong gravitational field induced by topographical masses poses another question, namely, which numerical method should be used to compute the gravitational effect of compensated masses. The traditional way is to expand Newton's kernel into a Taylor series, to take only a few first even series terms, and apply the fast Fourier transform to evaluate these terms numerically (e.g., Sideris et al., 1989). A questionable point of this procedure is whether the Taylor series converges or not, and if so, how many terms of the series should be taken into consideration to approximate the gravitational potential of topographical masses with a prescribed accuracy. In Chapter 5 we will find the condition under which the Taylor series converges and will attempt to estimate the size of individual series terms.

So far, all existing formulations of the direct and both indirect topographical effects have assumed that the topographical masses have a homogeneous mass density. The density was considered equal to a mean crustal value of $\varrho_0 = 2.67$ g/cm^3. This appears to be too coarse a model, especially in the vicinity of lakes, such as the Great Lakes in the central part of North America, because of the large difference between the water density and the mean crustal density ϱ_0. It is thus natural to ask whether the density contrast between lake water and surrounding rock is significant enough, and if it is, how this density inhomogeneity influences a precise geoid computation. This question is treated in Chapter 6. We will also consider another type of lateral density inhomogeneity occurring due to a individual geological formation and local geological factors; these may affect the topographical density as much as 10% to 20%. We estimate their impact on geoidal heights for a density pattern of geological structure beneath the Purcell Mountains (the part of the Canadian Rocky Mountains).

Vaníček and Kleusberg (1987) formulated the Stokes boundary-value problem for a higher-degree reference potential. They showed a number of advantages of such a formulation compared to the traditional formulation of the Stokes problem with a reference gravity field generated by a level ellipsoid. For instance, the truncation error of Stokes's integral applied to observed gravity anomaly data reduced to the reference gravity field is significantly smaller in the case when a higher-degree reference potential is employed (also Vaníček and Sjöberg, 1991).

In Chapter 7, we therefore reformulate the Stokes two-boundary-value problem for gravimetric determination of the geoid considering a satellite gravitational model as a reference. We intend to introduce a reference potential such that it does not depend on the way of compensation or condensation of topographical masses but only on the satellite reference model and on the gravitational field induced by topographical masses. The latter contribution to the reference potential is expressed in the form of an ellipsoidal harmonic series and the expansion coefficients are tabled numerically up to degree 20.

In Chapter 8, we investigate the stability of a discrete downward continuation problem for geoid determination when the surface gravity observations are harmonically continued from the Earth's surface to the geoid. The discrete form of Poisson's integral is used to set up the system of linear algebraic equations describing the problem. The posedness of the downward continuation problem is then tested by means of the eigenvalue analysis of this matrix. Numerically, we will treat the discrete downward continuation of gravity in a particularly rugged region of the Canadian Rocky Mountains.

The compensation of topographical masses is a possible way how to stabilize the downward continuation problem as the spectral contents of the gravity anomalies of compensated topographical masses may significantly differ from that of the original free-air gravity anomalies. Using again surface gravity data from the Canadian Rocky Mountains, we will investigate the efficiency of highly idealized compensation models, namely the Airy-Heiskanen model, the Pratt-Hayford model and Helmert's 2nd condensation technique, to dampen high-frequency oscillations of the free-air gravity anomalies.

Stokes's integral for the gravimetric determination of the geoid requires, besides other assumptions, that the gravity anomalies are referred to a sphere. According to Heiskanen and Moritz (1967, sect.2-14), the absolute error introduced by this spherical approximation does not exceed 1 m in terms of geoidal heights. A simple error analysis reveals that this error can even be larger and it may reach several metres. Such an error is unacceptable at a time where a 'decimetre geoid' is the target. In the Chapter 9, we aim at a solution to the Stokes problem for gravity anomalies distributed on an ellipsoid of revolution. The ellipsoidal approximation of the geoid reflects the reality much better than a spherical approximation since the actual shape of the geoid deviates from a level ellipsoid of revolution by 100 m at most. Treating the geoid as a level ellipsoid of revolution in Stokes's two-boundary-value problem may cause the absolute error of at most 2 cm in geoidal heights.

Green's function to the external Dirichlet boundary-value problem for the Laplace equation with data distributed on an ellipsoid of revolution is constructed in Chapter 10. The ellipsoidal Poisson kernel describing the effect of the ellipticity of the boundary on the solution to this boundary-value problem will be expressed in $O(e_0^2)$-approximation as a finite sum of elementary functions which describe analytically the behaviour of ellipsoidal Poisson kernel at the singular point $\psi = 0$. We intend to demonstrate that the degree of singularity of the ellipsoidal Poisson kernel in the vicinity of its singular point is of the same degree as that of the original spherical Poisson kernel.

In Chapter 11, we investigate the boundary-value problem of Stokes's type with ellipsoidal corrections in the boundary condition for anomalous gravity. Green's function approach enables us to avoid applying an iterative approach to solve this type of boundary-value problem since the solution is determined in one step by computing a Stokes-type integral. The question on the convergency of an iterative scheme that has been recommended so far is thus irrelevant.

Spherical harmonic coefficients of the Earth's external gravity field represent important characteristics of the Earth. Satellite tracking data, land-borne gravity observations and altimeter data must be combined to determine these coefficients with a sufficient reliability and resolution. This altimetry-gravimetry problem is governed by the Laplace equation in the space outside the Earth with a mixed type of boundary conditions. It has been proved that the classical least-squares solution to the *continuous* altimetry-gravimetry problem is not stable. On the contrary, Chapter 12 demonstrates that the least-squares solution to the *discrete* altimetry-gravimetry problem does not fail and is stable under satisfying certain conditions.

Most of the theoretical contributions described herein have already been published in the open literature, or the manuscripts describing the contributions have been either accepted or submitted for publication. I believe that this represents the best reviewing process for any research because the reviewing is done by an international group of referees. Thus, I refer to these papers wherever appropriate.