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# Three-space Problems in Banach Space Theory



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# Foreword

A three-space problem, in the Banach space setting, has the form: Let  $P$  be a Banach space property, let  $Y$  be a subspace of  $X$  and let  $X/Y$  be the corresponding quotient space. Is it true that  $X$  has property  $P$  when  $Y$  and  $X/Y$  have property  $P$ ? If the answer is positive then  $P$  is said to be a *three-space property*. We shall often shorten "three-space" to *3SP* (see below).

Three-space questions face the general problem of the structure of arbitrary subspaces and quotients of Banach spaces. Nowadays, it is clear that only Hilbert spaces can be labelled as simple: all subspaces are complemented Hilbert spaces and all quotients are Hilbert spaces. Other spaces may contain (and usually do) bizarre uncomplemented subspaces and/or quotients. The point of view of *3SP* problems is to consider the structure of a Banach space contemplating *as a whole* the structure of subspaces and the quotients they produce. Nevertheless, as it is noted in [136], "it may happen that a Banach space with quite complicated structure may possess nice factors through nice subspaces." When a property turns out to be a *3SP* property, it means that, despite the complexity that the combination of subspaces and quotients may have, the structure related with  $P$  is maintained. Incidentally, this gives a way to prove that a space possesses property  $P$ .

In the study of a *3SP* question it is interesting, when the answer is affirmative, to uncover the structure behind the proof; for instance, *being finite-dimensional*, *reflexive*, or *no containing  $l_1$*  are all *3SP* properties for essentially the same reason: the possibility of a lifting for the different types of sequences considered (resp. convergent, weakly convergent or weakly Cauchy). When it is not, the construction of a counterexample is more often than not a rewarding task. Also, there is the general question of how to construct the "middle" space starting with two spaces having  $P$ : one space  $Y$  that plays the role of the subspace and other  $Z$  that plays the role of the quotient. The simplest form to do this is making their direct sum  $Y \oplus Z$ . In direct sums the factor spaces appear complemented. Spaces admitting  $Y$  as a non-necessarily complemented subspace and  $Z$  as the corresponding quotient are called "twisted sums." The interesting general problem of how to construct twisted sums is considered, at an elementary level, in Chapters 1 and 3. This problem spreads into many ramifications: categorical methods, semi  $L$ -summands, inductive limits, construction of  $C(K)$  spaces, interpolation theory, etc, each of them contemplating a different aspect of the problem.

Conversely, a negative answer to a *3SP* question is also of interest because it

must involve a new construction of a space or an operator (usually the quotient map). It is then no surprise that some *3SP* examples can be translated into examples of pathological operators; this topic has not been sufficiently exploited in the literature, reason for which we present an overall view in an appendix to Chapter 2.

The origin of *3SP* problems can be traced back to a problem of Palais: If  $Y$  and  $X/Y$  are Hilbert spaces, has  $X$  to be isomorphic to a Hilbert space? Chronologically, *3SP* results did not formally appear until middle seventies, although Krein and Smullian's proof that reflexivity is a *3SP* property came around 1940. Counterexamples appeared with Enflo, Lindenstrauss and Pisier's solution to Palais problem (1975), while methods can be seriously considered after Jarchow's paper (1984). In this case, however, methods and counterexamples can be thought of as two different approaches to the same problem: some counterexamples provide a method, and methods sometimes provide a counterexample. Examples of counterexamples that provide methods could be the Kalton and Peck solution to Palais problem, which gave birth to the beautiful theory of quasi-linear maps; or Lusky's proofs that every Banach space  $X$  containing  $c_0$  admits a subspace  $Y$  such that both  $Y$  and  $X/Y$  have a basis.

Currently it is a standard question to ask for the *3SP* character of a new property. For this reason we devoted Chapter 2 to explain methods to obtain *3SP* properties. These methods include (and, we hope, cast some light on) some classical topics such as lifting results or factorization of operators. The counterexamples can be found in chapters 3 to 7 collected in what could be seen as something like a zoo.

Thus, we propose the reader a guided tour through that zoo. These notes exhibit alive all (all?) specimens of *3SP* problems that have been treated in the literature and which freely lived disseminated in many research papers, their natural habitat since they scarcely appeared in books. They have been patiently hunted and captured, carefully carried over (the common features of counterexamples have been emphasized), neatly polished (we often give simpler proofs) and classified following their nature (the proofs were unified through several general methods). There are also many new results and open problems.

The notation we follow is rather standard except in one point: the abbreviation *3SP* for "three-space." Although it is not standard, it is clear, clean and direct; started with our e-mail messages and has achieved some success.

We have intended to give an essentially self-contained exposition of *3SP* problems in the context of Banach spaces. There is a vast unexplored land of *3SP* problems in other contexts (locally convex spaces, Banach algebras,...) that we shall not consider. All properties appearing in the book are defined and their basic

relationships stated with appropriate references. Thus, some bounds about where to stop had to be imposed: things directly involved with  $3SP$  questions appear in detail; other results shall be just cited when they have already appeared in books. The word **Theorem** is reserved for results having the form: property  $P$  is (or is not) a  $3SP$  property. We made an effort to keep the reader informed about who did exactly what; this information is often given during the introductory comments although, in some cases, even conversations with the authors could not completely clarify where the ideas came from, and so we give our interpretation. About the problems and questions scattered through the text, we can only say that, to the best of our knowledge, they are open; some appear certainly to be difficult, some could even be easy, all seem interesting.

The background required to read these notes is a course in functional analysis and some familiarity with modern Banach space methods, as can be obtained from, e.g., J. Diestel's *Sequences and Series in Banach Spaces*. In order that the book can be used as a reference text we have included a summary, in alphabetical order, containing four entries for each property: 1) name; 2) yes, not, or open, for the corresponding  $3SP$  problem; 3) general method of proof, counterexample or additional information; 4) location of the result in the book.

Many people provided us with a constant flow of preprints, information, questions or advice that kept the project alive. To all of them our gratitude flows back. Working friends Félix Cabello, Rafael Payá, Fernando Sánchez and David Yost wrote several new proofs for this book, tried to re-do some sections ... and we are sure they would have written the entire book ... had we left them the opportunity. Susan Dierolf, Ricardo García, Hans Jarchow, Mar Jiménez, Antonio Martínez-Abejón, Mikhail Ostrovskii and Cristina Pérez made useful suggestions. Moreover, thanks should go to the colleagues of the Mathematics Department of the Universidad de Extremadura and of the University of Cantabria for providing the natural conditions for working. The topic of the book was discussed with colleagues and lectured on at several meetings during these three long years of elaboration: the Curso de Verano of the Universidad de Cantabria en Laredo; the Analysis Seminar at the Università di Bologna, organized by Piero Papini; the Conference on Functional Analysis at Camigliatello, organized by Antonio Carbone and Giuseppe Marino; the Conference on Function Spaces at the Universidad Complutense de Madrid, organized by José Luis LLavona and Angeles Prieto; and the 2<sup>nd</sup> Congress on Banach spaces at Badajoz. Thanks are due in each case to the organizers and supporting institutions.

**Beloved Boojums and the Bellman.** The first author thanks the students of the Analysis seminar who attended a course on  $3SP$  problems, and

especially to F. Arranz "Curro" who arranged a place for some invention. During this time, what started as a harmless photographic safari became a frumious hunting, sometimes galumphing. Perhaps no one has described it better than Lewis Carroll in *The Hunting of the Snark*; we suggest a careful reading of it to anyone interested in mathematical research. For instance, take the last but one stanza of the third fit.

I engage with the Snark - every night after dark -  
    In a dreamy delirious fight:  
I serve it with greens in those shadowy scenes,  
    And use it for striking a light.

On a higher ground, it seems to me that my mother and my sister have been working almost as much as I did to carry this book through. Without their supporting hand and their endless patience I would probably have softly and suddenly vanished away. This not being so, the real *3SP* problem for me has been then to combine the space of personal life, that of my University duties and the space required to make the book. Luna was the (precious) kernel that made this sequence exact. And one cannot leave unmentioned the exact sequence of our dogs: Suerte, Burbujas and Schwarz –which is my sister's dog, but anyway. Luna's mother and father, Javier Blanco, Antonio Hinchado (as the Banker) and Paco Pons (as the Booker) made a nice crew that helped to mix the bowsprit with the rudder sometimes.

The manuscript was patiently typed by Luna Blanco. Financial support came in part from the DGICYT (Spain) Project PB94-1052.

Although Bellman [51] affirms "*everything I say three times is true*" that does not necessarily apply to everything *we* say about three-spaces.



# Contents

<b>Foreword</b> . . . . .	v
<b>Contents</b> . . . . .	ix
<b>1. Three-space constructions</b> . . . . .	1
Short exact sequences . . . . .	2
The short exact sequence induced by the pull-back square . . . . .	4
The short exact sequence induced by the push-out square . . . . .	5
Extensions of Banach spaces . . . . .	8
Twisted sums of Banach spaces . . . . .	12
Snarked sums of Banach spaces . . . . .	20
Twisted sums and extensions . . . . .	27
Applications:	
Inductive limits of Banach spaces . . . . .	29
<b>Appendix. <math>L</math>-summands</b> . . . . .	34
Proper semi- $L$ -summands (34); Hahn-Banach smooth spaces (40)	
<b>Appendix. Harte's problem and Taylor's spectrum</b> . . . . .	42
<b>2. Methods to obtain three-space ideals</b> . . . . .	45
Three-space ideals . . . . .	46
Obtaining new three-space ideals from old ones . . . . .	47
Obtaining three-space ideals . . . . .	53
Lifting of sequences . . . . .	57
A second look at liftings . . . . .	60
Lifting for operators . . . . .	62
Three-space properties and semigroups of operators . . . . .	64
Jarchow's factorization method . . . . .	65
Incomparability classes . . . . .	66
$P$ -by- $Q$ and $P \oplus Q$ properties . . . . .	68
<b>Appendix. Applications of three-space results to the opening</b> . . . . .	71
The gap (or opening) between subspaces (71); Perturbation of operators (79)	

<b>3. Classical Banach spaces</b> . . . . .	81
The three-space problem for Hilbert spaces . . . . .	81
The approach of Enflo, Lindenstrauss and Pisier (82) The approach of Kalton and Peck (88); Weak Hilbert spaces (94)	
Three-space problems related to $l_p$ spaces . . . . .	95
Not containing $l_p$ ; $l_p$ saturated spaces (96); local properties (98)	
The three-space problem for $\mathcal{L}_p$ -spaces . . . . .	99
Three-space problems for $L_p$ -spaces . . . . .	100
Three-space problems for $C(K)$ spaces . . . . .	103
<b>Appendix. Banach spaces not isomorphic to their square</b> . . . . .	106
<b>Appendix. Dual Banach spaces</b> . . . . .	110
<b>4. Topological properties of Banach spaces</b> . . . . .	113
Reflexivity . . . . .	116
Quasi-reflexivity . . . . .	116
Co-reflexivity . . . . .	117
Somewhat reflexivity . . . . .	117
Super-reflexivity . . . . .	117
$W_p$ and $p$ -Banach-Saks properties . . . . .	118
Weak sequential completeness . . . . .	122
Weak* sequential compactness . . . . .	123
Mazur property or $d$ -completeness . . . . .	125
Weakly compactly generated spaces . . . . .	125
An intermission: proof of a theorem of Valdivia (130)	
Asplund spaces . . . . .	131
The Plichko-Valdivia property . . . . .	132
Polish spaces . . . . .	133
Cech completeness . . . . .	134
Between $WCG$ and $PRI$ . . . . .	137
$\sigma$ -fragmentability . . . . .	139
Point of continuity properties . . . . .	145
Property (C) of Pol . . . . .	150
GSG spaces . . . . .	152
<b>Appendix. Polynomial properties</b> . . . . .	154
Weakly sequentially continuous polynomials (154); Spaces admitting a separating polynomial (155)	
<b>5. Geometrical Properties</b> . . . . .	156
Type and cotype . . . . .	157
Basic results on renormings and differentiability of the norm . . . . .	159

Fréchet and $C^k$ -smooth renorming	161
Rough norms and Asplund spaces	164
Gâteaux smooth renorming	165
Weak Asplund spaces	168
$D$ -spaces	169
$w^*$ - $G_\delta$ extreme points	170
Measure compact spaces	170
Uniformly convex and locally uniformly convex renorming	171
$LUR$ renorming and fragmentability by slices	173
Mazur intersection property	176
Szlenk index	178
Midpoint $LUR$ renorming	180
The Krein-Milman property	181
Asymptotic norming properties	181
Strictly convex renorming	183
$\lambda$ -property	187
Octahedral norms	188
Kadec and Kadec-Klee norms	189
Ranges of vector measures	190
The U.M.D. property	192
Mazur rotations problem	193
Never condensed spaces	194
<b>6. Homological properties</b>	196
$C_p$ properties, $1 \leq p \leq \infty$	196
Orlicz properties	197
Grothendieck's theorem	200
Hilbert-Schmidt spaces	201
Radon-Nikodym properties	201
The Analytical $RNP$ (202); The Near $RNP$ (202)	
Dunford-Pettis properties	203
The hereditary Dunford-Pettis property (204); The "hereditary by quotients" Dunford-Pettis property (207); The Dunford-Pettis property of order $p$ , $1 < p \leq \infty$ . (207); Reciprocal Dunford-Pettis property (208); Somewhat $DP$ property (208); Surjective Dunford-Pettis property (209); Dieudonné property (210); Grothendieck property and its reciprocal (210); Other related properties (210)	
Covariant properties	211
The Gelfand-Phillips property	212
Properties $(u)$ , $(V)$ and $(V^*)$	216

Property ( $w$ ) . . . . .	219
Other properties . . . . .	220
Property ( $BD$ ) (220); Property $L$ (220); Dunford-Pettis sets are compact (220)	
<b>7. Approximation Properties</b> . . . . .	221
Spaces with Finite Dimensional Decompositions . . . . .	222
Spaces with a basis . . . . .	224
Positive results . . . . .	227
Skipped-blocking decompositions . . . . .	230
<b>Summary</b> . . . . .	232
<b>Bibliography</b> . . . . .	239
<b>Subject index</b> . . . . .	263