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# Three-space Problems in Banach Space Theory

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# Foreword

A three-space problem, in the Banach space setting, has the form: Let $P$ be a Banach space property, let $Y$ be a subspace of $X$ and let $X/Y$ be the corresponding quotient space. Is it true that $X$ has property $P$ when $Y$ and $X/Y$ have property $P$ ? If the answer is positive then $P$ is said to be a *three-space property*. We shall often shorten "three-space" to $3SP$ (see below).

Three-space questions face the general problem of the structure of arbitrary subspaces and quotients of Banach spaces. Nowadays, it is clear that only Hilbert spaces can be labelled as simple: all subspaces are complemented Hilbert spaces and all quotients are Hilbert spaces. Other spaces may contain (and usually do) bizarre uncomplemented subspaces and/or quotients. The point of view of $3SP$ problems is to consider the structure of a Banach space contemplating *as a whole* the structure of subspaces and the quotients they produce. Nevertheless, as it is noted in [136], "it may happen that a Banach space with quite complicated structure may possess nice factors through nice subspaces." When a property turns out to be a $3SP$ property, it means that, despite the complexity that the combination of subspaces and quotients may have, the structure related with $P$ is maintained. Incidentally, this gives a way to prove that a space possesses property $P$.

In the study of a $3SP$ question it is interesting, when the answer is affirmative, to uncover the structure behind the proof; for instance, *being finite-dimensional, reflexive,* or *no containing* $l_1$ are all $3SP$ properties for essentially the same reason: the possibility of a lifting for the different types of sequences considered (resp. convergent, weakly convergent or weakly Cauchy). When it is not, the construction of a counterexample is more often than not a rewarding task. Also, there is the general question of how to construct the "middle" space starting with two spaces having $P$: one space $Y$ that plays the role of the subspace and other $Z$ that plays the role of the quotient. The simplest form to do this is making their direct sum $Y\oplus Z$. In direct sums the factor spaces appear complemented. Spaces admitting $Y$ as a non-necessarily complemented subspace and $Z$ as the corresponding quotient are called "twisted sums." The interesting general problem of how to construct twisted sums is considered, at an elementary level, in Chapters 1 and 3. This problem spreads into many ramifications: categorical methods, semi $L$-summands, inductive limits, construction of $C(K)$ spaces, interpolation theory, etc, each of them contemplating a different aspect of the problem.

Conversely, a negative answer to a $3SP$ question is also of interest because it must involve a new construction of a space or an operator (usually the quotient map). It is then no surprise that some 3*SP* examples can be translated into examples of pathological operators; this topic has not been sufficiently exploited in the literature, reason for which we present an overall view in an appendix to Chapter 2.

The origin of 3*SP* problems can be traced back to a problem of Palais: If $Y$ and $X/Y$ are Hilbert spaces, has $X$ to be isomorphic to a Hilbert space? Chronologically, 3*SP* results did not formally appear until middle seventies, although Krein and Smullian's proof that reflexivity is a 3*SP* property came around 1940. Counterexamples appeared with Enflo, Lindenstrauss and Pisier's solution to Palais problem (1975), while methods can be seriously considered after Jarchow's paper (1984). In this case, however, methods and counterexamples can be thought of as two different approaches to the same problem: some counterexamples provide a method, and methods sometimes provide a counterexample. Examples of counterexamples that provide methods could be the Kalton and Peck solution to Palais problem, which gave birth to the beautiful theory of quasi-linear maps; or Lusky's proofs that every Banach space $X$ containing $c_0$ admits a subspace $Y$ such that both $Y$ and $X/Y$ have a basis.

Currently it is a standard question to ask for the 3*SP* character of a new property. For this reason we devoted Chapter 2 to explain methods to obtain 3*SP* properties. These methods include (and, we hope, cast some light on) some classical topics such as lifting results or factorization of operators. The counterexamples can be found in chapters 3 to 7 collected in what could be seen as something like a zoo.

Thus, we propose the reader a guided tour through that zoo. These notes exhibit alive all (all?) specimens of 3*SP* problems that have been treated in the literature and which freely lived disseminated in many research papers, their natural habitat since they scarcely appeared in books. They have been patiently hunted and captured, carefully carried over (the common features of counterexamples have been emphasized), neatly polished (we often give simpler proofs) and classified following their nature (the proofs were unified through several general methods). There are also many new results and open problems.

The notation we follow is rather standard except in one point: the abbreviation 3*SP* for "three-space." Although it is not standard, it is clear, clean and direct; started with our e-mail messages and has achieved some success.

We have intended to give an essentially self-contained exposition of 3*SP* problems in the context of Banach spaces. There is a vast unexplored land of 3*SP* problems in other contexts (locally convex spaces, Banach algebras,...) that we shall not consider. All properties appearing in the book are defined and their basic relationships stated with appropriate references. Thus, some bounds about where to stop had to be imposed: things directly involved with $3SP$ questions appear in detail; other results shall be just cited when they have already appeared in books. The word **Theorem** is reserved for results having the form: property $P$ is (or is not) a $3SP$ property. We made an effort to keep the reader informed about who did exactly what; this information is often given during the introductory comments although, in some cases, even conversations with the authors could not completely clarify where the ideas came from, and so we give our interpretation. About the problems and questions scattered through the text, we can only say that, to the best of our knowledge, they are open; some appear certainly to be difficult, some could even be easy, all seem interesting.

The background required to read these notes is a course in functional analysis and some familiarity with modern Banach space methods, as can be obtained from, e.g., J. Diestel's *Sequences and Series in Banach Spaces*. In order that the book can be used as a reference text we have included a summary, in alphabetical order, containing four entries for each property: 1) name; 2) yes, not, or open, for the corresponding $3SP$ problem; 3) general method of proof, counterexample or additional information; 4) location of the result in the book.

Many people provided us with a constant flow of preprints, information, questions or advice that kept the project alive. To all of them our gratitude flows back. Working friends Félix Cabello, Rafael Payá, Fernando Sánchez and David Yost wrote several new proofs for this book, tried to re-do some sections ... and we are sure they would have written the entire book ... had we left them the opportunity. Susan Dierolf, Ricardo García, Hans Jarchow, Mar Jiménez, Antonio Martínez-Abejón, Mikhail Ostrovskii and Cristina Pérez made useful suggestions. Moreover, thanks should go to the colleagues of the Mathematics Department of the Universidad de Extremadura and of the University of Cantabria for providing the natural conditions for working. The topic of the book was discussed with colleagues and lectured on at several meetings during these three long years of elaboration: the Curso de Verano of the Universidad de Cantabria en Laredo; the Analysis Seminar at the Universitá di Bologna, organized by Piero Papini; the Conference on Functional Analysis at Camigliatello, organized by Antonio Carbone and Giuseppe Marino; the Conference on Function Spaces at the Universidad Complutense de Madrid, organized by José Luis LLavona and Angeles Prieto; and the 2nd Congress on Banach spaces at Badajoz. Thanks are due in each case to the organizers and supporting institutions.

**Beloved Boojums and the Bellman.** The first author thanks the students of the Analysis seminar who attended a course on $3SP$ problems, and especially to F. Arranz "Curro" who arranged a place for some invention. During this time, what started as a harmless photographic safari became a frumious hunting, sometimes galumphing. Perhaps no one has described it better than Lewis Carroll in *The Hunting of the Snark*; we suggest a careful reading of it to anyone interested in mathematical research. For instance, take the last but one stanza of the third fit.

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I engage with the Snark - every night after dark -
          In a dreamy delirious fight:
  I serve it with greens in those shadowy scenes,
        And use it for striking a light.
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On a higher ground, it seems to me that my mother and my sister have been working almost as much as I did to carry this book through. Without their supporting hand and their endless patience I would probably have softly and suddenly vanished away. This not being so, the real $3SP$ problem for me has been then to combine the space of personal life, that of my University duties and the space required to make the book. Luna was the (precious) kernel that made this sequence exact. And one cannot leave unmentioned the exact sequence of our dogs: Suerte, Burbujas and Schwarz —which is my sister's dog, but anyway. Luna's mother and father, Javier Blanco, Antonio Hinchado (as the Banker) and Paco Pons (as the Booker) made a nice crew that helped to mix the bowsprit with the rudder sometimes.

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Although Bellman [51] affirms *"everything I say three times is true"* that does not necessarily apply to everything *we* say about three-spaces.

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