

# Lecture Notes in Mathematics

1693

Editors:

A. Dold, Heidelberg

F. Takens, Groningen

B. Teissier, Paris

**Springer**

*Berlin*

*Heidelberg*

*New York*

*Barcelona*

*Budapest*

*Hong Kong*

*London*

*Milan*

*Paris*

*Singapore*

*Tokyo*

Stephen Simons

# Minimax and Monotonicity



Springer

Author

Stephen Simons  
Department of Mathematics  
University of California  
Santa Barbara  
CA 93106-3080, USA  
simons@math.ucsb.edu

Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

**Simons, Stephen:**

**Minimax and monotonicity / Stephen Simons. - Berlin ; Heidelberg ; New York ; London ; Paris ; Tokyo ; Hong Kong ; Barcelona ; Budapest : Springer, 1998 (Lecture notes in mathematics ; 1693)  
ISBN 3-540-64755-4**

Mathematics Subject Classification (1991): 47H05, 47H04, 46B10, 49J35, 47N10

ISSN 0075-8434

ISBN 3-540-64755-4 Springer-Verlag Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting, reproduction on microfilms or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1998  
Printed in Germany

Typesetting: Camera-ready T<sub>E</sub>X output by the author

SPIN: 10649937      46/3143-543210 - Printed on acid-free paper

*For Jacqueline*

# Preface

These notes had their genesis in three hours of lectures that were given in a “School” on minimax theorems that was held in Erice, Sicily in September — October, 1996. This was followed by an expanded version in five hours of lectures at the “Spring School” on Banach spaces in Paseky in the Czech Republic in April, 1997 which was followed, in turn, by an even more expanded version in ten hours of lectures at the University of Toulouse, France in May – June, 1997.

The lectures were initially conceived as three isolated applications of minimax theorems to the theory of monotone multifunctions. With each successive iteration, the emphasis gradually shifted to an examination of the “minimax technique”, a method for proving the existence of continuous linear functionals on a Banach space, and to the relationship between this technique and monotone multifunctions. To this was finally added an attempt to collect together the results that have been proved on monotone and maximal monotone multifunctions on Banach spaces in recent years, and organize them into a coherent theory.

I would like to thank many people for their help and encouragement during the various stages of this project. I would first like to thank Biagio Ricceri for inviting me to Erice, Jaroslav Lukes, Jiri Kottas and Vaclav Zizler for inviting me to Paseky, and Jean-Baptiste Hiriart-Urruty for inviting me to Toulouse. I appreciate not only their excellent qualities as hosts, but also their grace and patience as audiences. Thanks are also due to Jonathan Borwein, Simon Fitzpatrick, Simeon Reich and Constantin Zalinescu for reading preliminary versions (or precursors) of these notes, and making many insightful comments and suggestions. I am especially grateful to Heinz Bauschke for reading a semi-final version of these notes from beginning to end, finding an amazing number of errors and ambiguities, and also for providing a number of excellent mathematical ideas. Last, but certainly not least, I would like to express my debt to Robert Phelps for his help and guidance all through this project. I appreciate his dogged insistence that I should try and make these notes as readable as possible. I would also like to acknowledge that his “Prague and Paseky” notes (which have been available electronically for several years) have been a source of inspiration.

VIII Preface

Of course, despite all the excellent efforts of the people mentioned above, these notes doubtless still contain errors and ambiguities, and also doubtless have other stylistic shortcomings. At any rate, I hope that there are not too many of these. Those that do exist are entirely my fault.

Stephen Simons  
May 28, 1998  
Santa Barbara  
California

# Table of Contents

Introduction .....	1
--------------------	---

---

## Chapter I. Functional analytic preliminaries

---

1. The Hahn–Banach and Mazur–Orlicz theorems .....	13
2. Convex, concave and affine functions .....	15
3. The minimax theorem .....	16
4. The dual and bidual of a Banach space .....	18
5. The minimax criterion for weak compactness in a Banach space .....	21
6. Four examples of the “minimax technique” — Fenchel duality .....	23
7. The perfect square trick and the $fg$ -theorem .....	27

---

## Chapter II. Multifunctions

---

8. Multifunctions, monotonicity and maximality .....	29
9. The “big convexification” .....	32
10. Criteria for maximal monotonicity in reflexive spaces ...	34
11. Monotone multifunctions with bounded range .....	40

---

## Chapter III. A digression into convex analysis

---

12. Surrounding sets and the dom lemma .....	43
13. The dom–dom lemma .....	45
14. The dom–dom lemma and the Attouch–Brézis condition	49

---

**Chapter IV. General monotone multifunctions**


---

15. Two convex functions determined by a multifunction . . .	53
16. Maximal monotonicity and closed convex sets . . . . .	57
17. A general local boundedness theorem . . . . .	63
18. The six set theorem and the nine set theorem . . . . .	64
19. The range of a sum . . . . .	70

---

**Chapter V. The sum problem for reflexive spaces**


---

20. The maximal monotonicity of a sum . . . . .	75
21. The dom–dom constraint qualification . . . . .	81
22. The six set and the nine set theorems for pairs of multifunctions . . . . .	84
23. The equivalence of six constraint qualifications — twice .	86
24. The Brézis–Crandall–Pazy condition . . . . .	89

---

**Chapter VI. Special maximal monotone multifunctions**


---

25. Subclasses of the maximal monotone multifunctions . . . .	97
26. The sum problem and the closure of the domain . . . . .	101
27. The closure of the range . . . . .	104

---

**Chapter VII. Subdifferentials**


---

28. The subdifferential of a sum . . . . .	111
29. Subdifferentials are maximal monotone . . . . .	113
30. Subdifferentials are of type (FP) . . . . .	118
31. Subdifferentials are of type (FPV) . . . . .	120
32. Subdifferentials are strongly maximal monotone . . . . .	123
33. The biconjugate of a pointwise maximum . . . . .	129
34. Biconjugate topologies on the bidual . . . . .	132

<b>35. Subdifferentials are maximal monotone of type (D), and more</b> .....	138
--	-----

---

**Chapter VIII. Discontinuous positive linear operators**

---

<b>36. A criterion for maximality</b> .....	141
<b>37. A sum theorem</b> .....	143
<b>38. Discontinuous positive linear operators and the “six subclasses”</b> .....	145

---

**Chapter IX. The sum problem for general Banach spaces**

---

<b>39. Introduction</b> .....	153
<b>40. Multifunctions with full domain</b> .....	153
<b>41. Sums with normality maps</b> .....	156
<b>42. Sums with linear maps</b> .....	160

---

**Chapter X. Open problems** .....

---

<b>References</b> .....	165
<b>Subject index</b> .....	169
<b>Symbol index</b> .....	171