# Part II
# First Steps Toward Conformal Field Theory

The term “conformal field theory” stands for a variety of different formulations and slightly different structures. The aim of the second part of these notes is to describe some of these formulations and structures and thereby contribute to answering the question of what conformal field theory is.

Conformal field theories are best described either by the way they appear and are constructed or by properties and axioms which provide classes of conformal field theories. The most common theories by examples are

- free bosons or fermions ($\sigma$-models on a torus),
- WZW-models[1] for compact Lie groups and gauged WZW-models,
- coset and orbifold constructions of WZW-models.

Systematic descriptions of conformal field theory emphasizing the fundamental structures and properties comprise

- various combinatorical approaches like the axioms of Moore–Seiberg [MS89], Friedan–Shenker [FS87], or Segal [Seg88a].
- the Osterwalder–Schrader axioms with conformal invariance [FFK89],
- the vertex algebras or chiral algebras [BD04*] as their generalizations,

A common feature and essential point of all these approaches to conformal field theory is the appearance of representations of the Virasoro algebra which play a central role. The simple reason for this major role of the Virasoro is based on the fact that the elements of the Virasoro algebra are symmetries of the quantum system and these elements are regarded as the most important observables in conformal field theory. In this context the generators $L_n$ can be compared in their physical significance to the momentum or angular momentum in conventional one-particle quantum mechanics.

Since the Witt algebra $\mathsf{W}$ is a generating subalgebra of the infinitesimal classical conformal transformations of the Minkowski plane in each of the two light cone variables (cf. Corollary 2.15 and Sect. 5.1), the set of all observables of conformal field theory contains the direct product $\mathsf{Vir} \times \overline{\mathsf{Vir}}$ of two copies of the Virasoro algebra. (Note that after quantization, the Witt algebra has to be replaced by its nontrivial central extension, the Virasoro algebra $\mathsf{Vir}$, cf. Chaps. 3 and 4.) In general, one assumes the full set $\mathscr{A}_{\text{tot}}$ of observables to form an algebra which decomposes into a direct product of algebras $\mathscr{A} \times \mathscr{A}'$ containing the Virasoro algebras $\mathsf{Vir} \subset \mathscr{A}, \overline{\mathsf{Vir}} \subset \mathscr{A}'$. The two components of the full algebra of observables are called chiral halves or holomorphic/antiholomorphic or similar.

As a consequence of the product structure, for many purposes one can restrict the investigations to one “chiral half” of the theory in such a way that only $\mathsf{Vir} \subset \mathscr{A}$ resp. $\overline{\mathsf{Vir}} \subset \mathscr{A}'$ is studied. The restrictions to one chiral half requires among other things to regard the light cone variables $t_+$ and $t_-$ as completely independent variables, and, in the same way, the complex variables $z$ and $\overline{z}$ as completely independent. The identification of $\overline{z}$ with the complex conjugate only takes place when the two chiral halves of the conformal field theory are combined.

[1] WZW = Wess–Zumino–Witten

Restricting now to one chiral half $\mathscr{A}$ and, furthermore, restricting to the subalgebra $\mathsf{Vir}$ we are led, first of all, to study the representations of the Virasoro algebra.

In a certain way one could claim now that conformal field theory is the representation theory of the Virasoro algebra and of certain algebras (namely chiral algebras) containing the Virasoro algebra. Therefore, in this second part of the notes we first describe the representations of the Virasoro algebra (Chap. 6) and explain as an example how the quantization of strings leads to a representation of the Virasoro algebra (Chap. 7). Next we discuss the axiomatic approach to quantum field theory according to Wightman as well as the Euclidean version according to Osterwalder–Schrader (Chap. 8) and treat the case of two-dimensional conformal field theory in a separate chapter (Chap. 9). In Chap. 10 we connect all these with the theory of vertex algebras, and in Chap. 11 we present as an example of an application of conformal field theory to complex algebraic geometry the Verlinde formula in the context of holomorphic vector bundles and moduli spaces.