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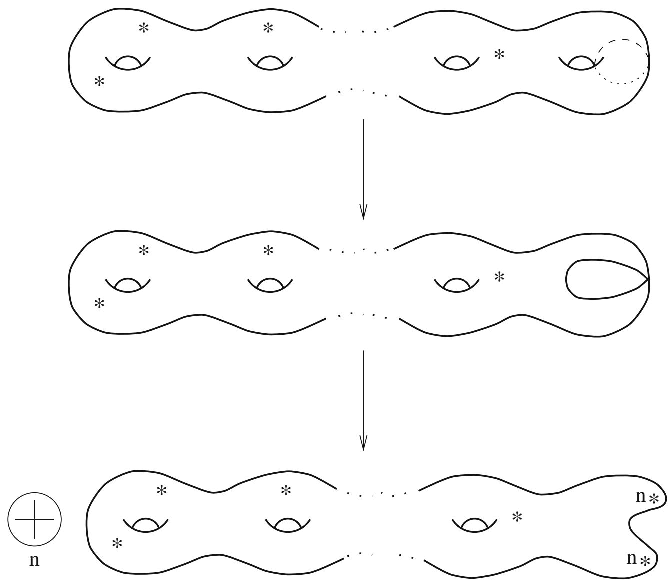
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A Mathematical Introduction to Conformal Field Theory

Second Edition

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To Barbara

Preface to the Second Edition

The second edition of these notes has been completely rewritten and substantially expanded with the intention not only to improve the use of the book as an introductory text to conformal field theory, but also to get in contact with some recent developments. In this way we take a number of remarks and contributions by readers of the first edition into consideration who appreciated the rather detailed and self-contained exposition in the first part of the notes but asked for more details for the second part. The enlarged edition also reflects experiences made in seminars on the subject.

The interest in conformal field theory has grown during the last 10 years and several texts and monographs reflecting different aspects of the field have been published as, e.g., the detailed physics-oriented introduction of Di Francesco, Mathieu, and Sénéchal [DMS96*],[1] the treatment of conformal field theories as vertex algebras by Kac [Kac98*], the development of conformal field theory in the context of algebraic geometry as in Frenkel and Ben-Zvi [BF01*] and more general by Beilinson and Drinfeld [BD04*]. There is also the comprehensive collection of articles by Deligne, Freed, Witten, and others in [Del99*] aiming to give an introduction to strings and quantum field theory for mathematicians where conformal field theory is one of the main parts of the text. The present expanded notes complement these publications by giving an elementary and comparatively short mathematics-oriented introduction focusing on some main principles.

The notes consist of 11 chapters organized as before in two parts. The main changes are two new chapters, Chap. 8 on Wightman's axioms for quantum field theory and Chap. 10 on vertex algebras, as well as the incorporation of several new statements, examples, and remarks throughout the text. The volume of the text of the new edition has doubled. Half of this expansion is due to the two new chapters.

We have included an exposition of Wightman's axioms into the notes because the axioms demonstrate in a convincing manner how a consistent quantum field theory in principle should be formulated even regarding the fact that no four-dimensional model with properly interacting fields satisfying the axioms is known to date. We investigate in Chap. 8 the axioms in their different appearances as postulates on operator-valued distributions in the relativistic case as well as postulates on the

[1] The "*" indicates that the respective reference has been added to the References in the second edition of these notes.

corresponding correlation functions on Minkowski and on Euclidean spaces. The presentation of the axioms serves as a preparation and motivation for Chap. 9 as well as for Chap. 10.

Chapter 9 deals with an axiomatic approach to two-dimensional conformal field theory. In comparison to the first edition we have added the conformal Ward identities, the state field correspondence, and some changes with respect to the presentation of the operator product expansion. The concepts and methods in this chapter were quite isolated in the first edition, and they can now be understood in the context of Wightman's axioms in its various forms and they also can be linked to the theory of vertex algebras.

Vertex algebras have turned out to be extremely useful in many areas of mathematics and physics, and they have become the main language of two-dimensional conformal field theory in the meantime. Therefore, the new Chap. 10 in these notes provides a presentation of basic concepts and methods of vertex algebras together with some examples. In this way, a number of manipulations in Chap. 9 are explained again, and the whole presentation of vertex algebras in these notes can be understood as a kind of formal and algebraic continuation of the axiomatic treatment of conformal field theory.

Furthermore, many new examples have been included which appear at several places in these notes and may serve as a link between the different viewpoints (for instance, the Heisenberg algebra H as an example of a central extension of Lie algebras in Chap. 4, as a symmetry algebra in the context of quantization of strings in Chap. 7, and as a first main example of a vertex algebra in Chap. 10). Similarly, Kac–Moody algebras are introduced, as well as the free bosonic field and the restricted unitary group in the context of quantum electrodynamics. Several of the elementary but important statements of the first edition have been explained in greater detail, for instance, the fact that the conformal groups of the Euclidean spaces are finite dimensional, even in the two-dimensional case, the fact that there does not exist a complex Virasoro group and that the unitary group $\mathrm{U}(\mathbb{H})$ of an infinite-dimensional Hilbert space $\mathbb{H}$ is a topological group in the strong topology.

Moreover, several new statements have been included, for instance, about a detailed description of some classical groups, about the quantization of the harmonic oscillator and about general principles used throughout the notes as, for instance, the construction of representations of Lie algebras as induced representations or the use of semidirect products.

The general concept of presenting a rather brief and at the same time rigorous introduction to conformal field theory is maintained in this second edition as well as the division of the notes in two parts of a different nature: The first is quite elementary and detailed, whereas the second part requires more mathematical prerequisites, in particular, from functional analysis, complex analysis, and complex algebraic geometry.

Due to the complexity of the treatment of Wightman's axioms in the second part of the notes not all results are proven, but there are many more proofs in the second part than in the original edition. In particular, the chapter on vertex algebras is self-contained.

The final chapter on the Verlinde formula in the context of algebraic geometry, which is now Chap. 11, has nearly not been changed except for a comment on fusion rings and on the connection of the Verlinde algebra with twisted K-theory recently discovered by Freed, Hopkins, and Teleman [FHT03*].

In a brief appendix we mention further developments with respect to boundary conformal field theory, to stochastic Loewner evolution, and to modularity together with some references.

München, March 2008 *Martin Schottenloher*

References

BD04*. A. Beilinson and V. Drinfeld *Chiral Algebras*. AMS Colloquium Publications **51** AMS, Providence, RI, 2004. vii

BF01*. D. Ben-Zvi and E. Frenkel *Vertex Algebras and Algebraic Curves*. AMS, Providence, RI, 2001. vii

Del99*. P. Deligne et al. *Quantum Fields and Strings: A Course for Mathematicians I, II*. AMS, Providence, RI, 1999. vii

DMS96*. P. Di Francesco, P. Mathieu and D. Sénéchal. *Conformal Field Theory*. Springer-Verlag, 1996. vii

FHT03*. D. Freed, M. Hopkins, and C. Teleman. Loop groups and twisted K-theory III. `arXiv:math/0312155v3` (2003). ix

Kac98*. V. Kac. *Vertex Algebras for Beginners*. University Lecture Series **10**, AMS, Providencs, RI, 2^{nd} ed., 1998. vii

Preface to the First Edition

The present notes consist of two parts of approximately equal length. The first part gives an elementary, detailed, and self-contained mathematical exposition of classical conformal symmetry in n dimensions and its quantization in two-dimensions. Central extensions of Lie groups and Lie algebras are studied in order to explain the appearance of the Virasoro algebra in the quantization of two-dimensional conformal symmetry. The second part surveys some topics related to conformal field theory: the representation theory of the Virasoro algebra, some aspects of conformal symmetry in string theory, a set of axioms for a two-dimensional conformally invariant quantum field theory, and a mathematical interpretation of the Verlinde formula in the context of semi-stable holomorphic vector bundles on a Riemann surface. In contrast to the first part only few proofs are provided in this less elementary second part of the notes.

These notes constitute – except for corrections and supplements – a translation of the prepublication "Eine mathematische Einführung in die konforme Feldtheorie" in the preprint series *Hamburger Beiträge zur Mathematik*, Volume 38 (1995). The notes are based on a series of lectures I gave during November/December of 1994 while holding a *Gastdozentur* at the *Mathematisches Seminar der Universität Hamburg* and on similar lectures I gave at the *Université de Nice* during March/April 1995.

It is a pleasure to thank H. Brunke, R. Dick, A. Jochens, and P. Slodowy for various helpful comments and suggestions for corrections. Moreover, I want to thank A. Jochens for writing a first version of these notes and for carefully preparing the LaTeX file of an expanded English version. Finally, I would like to thank the Springer production team for their support.

Munich, September 1996 *Martin Schottenloher*

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Introduction

Conformal field theory in two dimensions has its roots in statistical physics (cf. [BPZ84] as a fundamental work and [Gin89] for an introduction) and it has close connections to string theory and other two-dimensional field theories in physics (cf., e.g., [LPSA94]). In particular, all massless fields are conformally invariant.

The special feature of conformal field theory in two dimensions is the existence of an infinite number of independent symmetries of the system, leading to corresponding invariants of motion which are also called conserved quantities. This is the content of Noether's theorem which states that a symmetry of a physical system given by a local one-parameter group or by an infinitesimal version thereof induces an invariant of motion of the system. Any collection of invariants of motion simplifies the system in question up to the possibility of obtaining a complete solution. For instance, in a typical system of classical mechanics an invariant of motion reduces the number of degrees of freedom. If the original phase space has dimension $2n$ the application of an invariant of motion leads to a system with a phase space of dimension $2(n-1)$. In this way, an independent set of n invariants of motion can lead to a zero-dimensional phase space that means, in general, to a complete solution.

Similarly, in the case of conformal field theory the invariants of motion which are induced by the infinitesimal conformal symmetries reduce the infinite dimensional system completely. As a consequence, the structure constants which determine the system can be calculated explicitly, at least in principle, and one obtains a complete solution. This is explained in Chap. 9, in particular in Proposition 9.12.

These symmetries in a conformal field theory can be understood as infinitesimal conformal symmetries of the Euclidean plane or, more generally, of surfaces with a conformal structure, that is Riemann surfaces. Since conformal transformations on an open subset U of the Euclidean plane are angle preserving, the conformal orientation-preserving transformations on U are holomorphic functions with respect to the natural complex structure induced by the identification of the Euclidean plane with the space $\mathbb{C}$ of complex numbers. As a consequence, there is a close connection between conformal field theory and function theory. A good portion of conformal field theory is formulated in terms of holomorphic functions using many results of function theory. On the other hand, this interrelation between conformal field theory and function theory yields remarkable results on moduli spaces of vector bundles

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over compact Riemann surfaces and therefore provides an interesting example of how physics can be applied to mathematics.

The original purpose of the lectures on which the present text is based was to describe and to explain the role the Virasoro algebra plays in the quantization of conformal symmetries in two dimensions. In view of the usual difficulties of a mathematician reading research articles or monographs on conformal field theory, it was an essential concern of the lectures not to rely on background knowledge of standard methods in physics. Instead, the aim was to try to present all necessary concepts and methods on a purely mathematical basis. This explains the adjective "mathematical" in the title of these notes. Another motivation was to discuss the sometimes confusing use of language by physicists, who for example emphasize that the group of holomorphic maps of the complex plane is infinite dimensional – which is not true. What is meant by this statement is that a certain Lie algebra closely related to conformal symmetry, namely the Witt algebra or its central extension, the Virasoro algebra, is infinite dimensional.

Clearly, with these objectives the lectures could hardly cover an essential part of actual conformal field theory. Indeed, in the course of the present text, conformally invariant quantum field theory does not appear before Chap. 6, which treats the representation theory of the Virasoro algebra as a first topic of conformal field theory. These notes should therefore be seen as a preparation for or as an introduction to conformal field theory for mathematicians focusing on some background material in geometry and algebra. Physicists may find the detailed investigation in Part I useful, where some elementary geometric and algebraic prerequisites for conformal field theory are studied, as well as the more advanced mathematical description of fundamental structures and principles in the context of quantum field theory in Part II.

In view of the above-mentioned tasks, it makes sense to start with a detailed description of the conformal transformations in arbitrary dimensions and for arbitrary signatures (Chap. 1) and to determine the associated conformal groups (Chap. 2) with the aid of the conformal compactification of spacetime. In particular, the conformal group of the Minkowski plane turns out to be infinite dimensional, it is essentially isomorphic to $\text{Diff}_+(\mathbb{S}^1) \times \text{Diff}_+(\mathbb{S}^1)$, while the conformal group of the Euclidean plane is finite-dimensional, it is the group of Möbius transformations isomorphic to $\text{SL}(2,\mathbb{C})/\{\pm 1\}$.

The next two chapters (Chaps. 3 and 4) are concerned with central extensions of groups and Lie algebras and their classification by cohomology. These two chapters contain several examples appearing in physics and mathematics. Central extensions are needed in physics, because the symmetry group of a quantized system usually is a central extension of (the universal covering of) the classical symmetry group, and in the same way the infinitesimal symmetry algebra of the quantum system is, in general, a central extension of the classical symmetry algebra.

Chapter 5 leads to the Virasoro algebra as the unique nontrivial central extension of the Witt algebra. The Witt algebra is the essential component of the classical infinitesimal conformal symmetry in two dimensions for the Euclidean plane as well as for the Minkowski plane. This concludes the first part of the text which is comparatively elementary except for some aspects in the examples.

The second part presents several different approaches to conformal field theory. We start this program with the representation theory of the Virasoro algebra including the Kac formula (Chap. 6) in order to describe the unitary representations.

In Chap. 7 we give an elementary introduction into the quantization of the bosonic string and explain how the conformal symmetry is present in classical and in quantized string theory. The quantization induces a natural representation of the Virasoro algebra on the Fock space of the Heisenberg algebra which is of interest in later considerations concerning examples of vertex algebras.

The next two chapters are dedicated to axiomatic quantum field theory. In Chap. 8 we provide an exposition of the relativistic case in any dimension by presenting the Wightman axioms for the field operators as well as the equivalent axioms for the correlation functions called Wightman distributions. The Wightman distributions are boundary values of holomorphic functions which can be continued analytically into a large domain in complexified spacetime and thereby provide the correlation functions of a Euclidean version of the axioms, the Osterwalder–Schrader axioms. In Chap. 9 we concentrate on the two-dimensional Euclidean case with conformal symmetry. We aim to present an axiomatic approach to conformal field theory along the suggestion of [FFK89] and the postulates of the groundbreaking paper of Belavin, Polyakov, and Zamolodchikov [BPZ84].

Many papers on conformal field theory nowadays use the language of vertex operators and vertex algebras. Chapter 10 gives a brief introduction to the basic concepts of vertex algebras and some fundamental results. Several concepts and constructions reappear in this chapter – sometimes in a slightly different form – so that one has a common view of the different approaches to conformal field theory presented in the preceding chapters.

Finally we discuss the Verlinde formula as an application of conformal field theory to mathematics (Chap. 11).

References

BPZ84. A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nucl. Phys.* **B 241** (1984), 333–380. 1, 3

FFK89. G. Felder, J. Fröhlich, and J. Keller. On the structure of unitary conformal field theory, I. Existence of conformal blocks. *Comm. Math. Phys.* **124** (1989), 417–463. 3, 88

Gin89. P. Ginsparg. Introduction to conformal field theory. *Fields, Strings and Critical Phenomena*, Les Houches 1988, Elsevier, Amsterdam 1989. 1

LPSA94. R. Langlands, P. Pouliot, and Y. Saint-Aubin. Conformal invariance in two-dimensional percolation. *Bull. Am. Math. Soc.* **30** (1994), 1–61. 1