Applied Mathematical Sciences

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Boundary Integral Equations

Springer

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To our families for their love and understanding

Preface

This book is devoted to the mathematical foundation of boundary integral equations. The combination of finite element analysis on the boundary with these equations has led to very efficient computational tools, the boundary element methods (see e. g., the authors [139] and Schanz and Steinbach (eds.) [267]). Although we do not deal with the boundary element discretizations in this book, the material presented here gives the mathematical foundation of these methods. In order to avoid over generalization we have confined ourselves to the treatment of elliptic boundary value problems.

The central idea of eliminating the field equations in the domain and reducing boundary value problems to equivalent equations only on the boundary requires the knowledge of corresponding fundamental solutions, and this idea has a long history dating back to the work of Green [107] and Gauss [95, 96]. Today the resulting boundary integral equations still serve as a major tool for the analysis and construction of solutions to boundary value problems.

As is well known, the reduction to equivalent boundary integral equations is by no means unique, and there are primarily two procedures for this reduction, the 'direct' and 'indirect' approaches. The direct procedure is based on Green's representation formula for solutions of the boundary value problem, whereas the indirect approach rests on an appropriate layer ansatz. In our presentation we concentrate on the direct approach although the corresponding analysis and basic properties of the boundary integral operators remain the same for the indirect approaches. Roughly speaking, one obtains two kinds of boundary integral equations with both procedures, those of the first kind and those of the second kind.

The basic mathematical properties that guarantee existence of solutions to the boundary integral equations and also stability and convergence analysis of corresponding numerical procedures hinge on Gårding inequalities for the boundary integral operators on appropriate function spaces. In addition, contraction properties allow the application of Carl Neumann's classical successive iteration procedure to a class of boundary integral equations of the second kind. It turns out that these basic features are intimately related to the variational forms of the underlying elliptic boundary value problems and the potential energies of their solution fields, allowing us to consider the boundary integral equations in the form of variational problems on the boundary manifold of the domain.

On the other hand, the Newton potentials as the inverses of the elliptic partial differential operators are particular pseudodifferential operators on the domain or in the Euclidean space. The boundary potentials (or Poisson operators) are just Newton potentials of distributions with support on the boundary manifold and the boundary integral operators are their traces there. Therefore, it is rather natural to consider the boundary integral operators as pseudodifferential operators on the boundary manifold. Indeed, most of the boundary integral operators in applications can be recast as such pseudodifferential operators provided that the boundary manifold is smooth enough.

With the application of boundary element methods in mind, where strong ellipticity is the basic concept for stability, convergence and error analysis of corresponding discretization methods for the boundary integral equations, we are most interested in establishing strong ellipticity in terms of Gårding's inequality for the variational formulation as well as strong ellipticity of the pseudodifferential operators generated by the boundary integral equations. The combination of both, namely the variational properties of the elliptic boundary value and transmission problems as well as the strongly elliptic pseudodifferential operators provides us with an efficient means to analyze a large class of elliptic boundary value problems.

This book contains 10 chapters and an appendix. For the reader's benefit, Figure 0.1 gives a sketch of the topics contained in this book. Chapters 1 through 4 present various examples and background information relevant to the premises of this book.

In Chapter 5, we discuss the variational formulation of boundary integral equations and their connection to the variational solution of associated boundary value or transmission problems. In particular, continuity and coerciveness properties of a rather large class of boundary integral equations are obtained, including those discussed in the first and second chapters. In Chapter 4, we collect basic properties of Sobolev spaces in the domain and their traces on the boundary, which are needed for the variational formulations in Chapter 5.

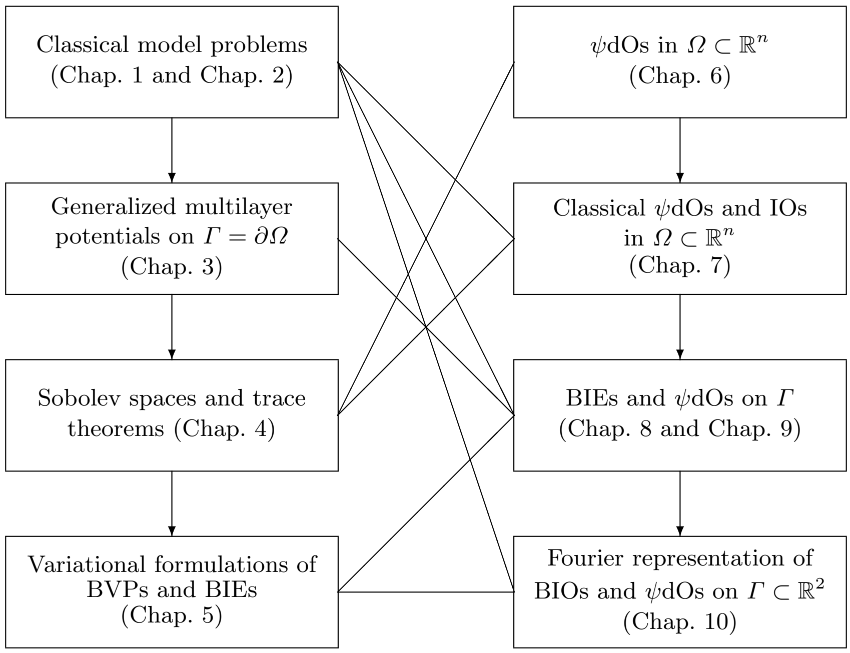
Chapter 6 presents an introduction to the basic theory of classical pseudodifferential operators. In particular, we present the construction of a parametrix for elliptic pseudodifferential operators in subdomains of $\mathbb{R}^n$. Moreover, we give an iterative procedure to find Levi functions of arbitrary order for general elliptic systems of partial differential equations. If the fundamental solution exists then Levi's method based on Levi functions allows its construction via an appropriate integral equation.

In Chapter 7, we show that every pseudodifferential operator is an Hadamard's finite part integral operator with integrable or nonintegrable kernel plus possibly a differential operator of the same order as that of the pseudodifferential operator in case of nonnegative integer order. In addition, we formulate the necessary and sufficient Tricomi conditions for the integral operator kernels to define pseudodifferential operators in the domain by using the asymptotic expansions of the symbols and those of pseudohomogeneous kernels. We close Chapter 7 with a presentation of the transformation formulae and invariance properties under the change of coordinates.

Chapter 8 is devoted to the relation between the classical pseudodifferential operators and boundary integral operators. For smooth boundaries, all of our examples in Chapters 1 and 2 of boundary integral operators belong to the class of classical pseudodifferential operators on compact manifolds having symbols of the rational type. If the corresponding class of pseudodifferential operators in the form of Newton potentials is applied to tensor product distributions with support on the boundary manifold, then they generate, in a natural way, boundary integral operators which again are classical pseudodifferential operators on the boundary manifold. Moreover, for these operators associated with regular elliptic boundary value problems, it turns out that the corresponding Hadamard's finite part integral operators are invariant under the change of coordinates, as considered in Chapter 3. This approach also provides the jump relations of the potentials. We obtain these properties by using only the Schwartz kernels of the boundary integral operators. However, these are covered by Boutet de Monvel's work in the 1960's on regular elliptic problems involving the transmission properties.

The last two chapters, 9 and 10, contain concrete examples of boundary integral equations in the framework of pseudodifferential operators on the boundary manifold. In Chapter 9, we provide explicit calculations of the symbols corresponding to typical boundary integral operators on closed surfaces in $\mathbb{R}^3$. If the fundamental solution is not available then the boundary value problem can still be reduced to a coupled system of domain and boundary integral equations. As an illustration we show that these coupled systems can be considered as some particular Green operators of the Boutet de Monvel algebra. In Chapter 10, the special features of Fourier series expansions of boundary integral operators on closed curves are exploited.

We conclude the book with a short Appendix on differential operators in local coordinates with minimal differentiability. Here, we avoid the explicit use of the normal vector field as employed in Hadamard's coordinates in Chapter 3. These local coordinates may also serve for a more detailed analysis for Lipschitz domains.



Abbreviations:

$\Omega \subset \mathbb{R}^n$	– *A given domain with compact boundary* Γ
BVPs	– *Boundary value problems*
BIEs	– *Boundary integral equations*
ψ*dOs*	– *Pseudodifferential operators*
IOs	– *Integral operators*
BIOs	– *Boundary integral operators*

Fig. 0.1. A schematic sketch of the topics and their relations

Our original plan was to finish this book project about 10 years ago. However, many new ideas and developments in boundary integral equation methods appeared during these years which we have attempted to incorporate. Nevertheless, we regret to say that the present book is by no means complete. For instance, we only slightly touch on the boundary integral operator methods involving Lipschitz boundaries which have recently become more important in engineering applications. We do hope that we have made a small step forward to bridge the gap between the theory of boundary integral equation methods and their applications. We further hope that this book will lead to better understanding of the underlying mathematical structure of these methods and will serve as a mathematical foundation of the boundary element methods.

In closing, we would also like to mention some other relevant books related to boundary integral methods such as the classical books on potential theory by Kellogg [155] and Günter [113], the mathematical books on boundary integral equations by Hackbusch [116], Jaswon and Symm [148], Kupradze [175, 176, 177], Schatz, Thomée and Wendland [268], Mikhlin [211, 212, 213], Nedelec [231, 234], Colton and Kress [47, 48], Mikhlin, Morozov and Paukshto [214], Mikhlin and Prössdorf [215], Dautray and Lions [60], Chen and Zhou [40], Gatica and Hsiao [93], Kress [172], McLean [203], Yu, De–hao [324], Steinbach [290], Freeden and Michel [83], Kohr and Pop [163], Sauter and Schwab [266], as well as the Encyclopedia articles by Maz'ya [202], Prössdorf [253], Agranovich [4] and the authors [141]. For engineering books on boundary integral equations, we suggest the books by Brebbia [23], Crouch and Starfield [57], Brebbia, Telles and Wrobel [24], Manolis and Beskos [197], Balaš, Sladek and Sladek [11], Pozrikidis [252], Power and Wrobel [251], Bonnet [18], Gaul, Kögel and Wagner [94].

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