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Karel Dekimpe

Almost-Bieberbach Groups: Affine and Polynomial Structures

Springer

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For Katleen, Charlotte and Sofie.

Preface

The reader taking a first glance at this monograph might have the (wrong) impression that a lot of topology/geometry is involved. Indeed, the objects we study in this book are a special kind of manifold, called the infra–nilmanifolds. This is a class of manifolds that can, and should, be viewed as a generalization of the flat Riemannian manifolds. However, the reader familiar with the theory of the flat Riemannian manifolds knows that such a manifold is completely determined by its fundamental group. Moreover, the groups that occur as such a fundamental group can be characterized in a purely algebraic way. More precisely, a group E is the fundamental group of a flat Riemannian manifold if and only if E is a finitely generated torsion free group containing a normal abelian subgroup of finite index. These groups are called Bieberbach groups. It follows that one can study the flat Riemannian manifolds in a purely algebraic way.

This group theoretical approach is also possible for the infra–nilmanifolds, which are obtained as a quotient space under the action of a group E on a simply connected nilpotent Lie group G, where E acts properly discontinuously and via isometries on G. (If G is abelian, then this quotient space is exactly a flat Riemannian manifold). The fundamental group of an infra–nilmanifold is referred to as an almost–Bieberbach group. It turns out that much of the theory of Bieberbach groups extends to the almost–Bieberbach groups. Thus for instance, a group E is the fundamental group of an infra–nilmanifold if and only if E is a finitely generated torsion free group containing a normal nilpotent subgroup of finite index.

The aim of this book is twofold:

1. I wish to explain and describe (in full detail) some of the most important group–theoretical properties of almost–Bieberbach groups.

I have the impression that the algebraic nature of almost–Bieberbach groups is far from well known, although many of their properties are just a straightforward generalization of the corresponding properties of the Bieberbach groups. On the other hand, I do not claim to be a specialist of Bieberbach (or more general crystallographic) groups and so a lot more of the theory of Bieberbach (crystallographic) groups still has to be generalized. I hope therefore that this book might stimulate the reader to help in this generalization.

2. I also felt there is a need for a detailled classification of all almost–Bieberbach groups in dimensions ≤ 4. We will see that an infra–nilmanifold is completely determined by its fundamental group. So my classification of almost–Bieberbach groups can also be viewed as a classification of all infra–nilmanifolds of dimensions ≤ 4. I myself use the tables of almost–Bieberbach groups not really as a classification but as an elaborated set of examples or "test cases" for new hypotheses. I hope that, one day, they can be of the same value to you too.

I tried to write this monograph both for topologists/geometers as for algebraists. Therefore, I made an effort to keep the prerequisites as low as possible. However, the reader should have at least an idea of what a Lie group is. Also, a little knowledge of the theory of covering spaces can be helpful now and then. From the algebraic point of view, I assume that the reader is fairly familiar with nilpotent groups and that he is acquainted with group extensions and its relation to cohomology of groups.

Although this work is divided into eight chapters, there are really three parts to distinguish.

1. In the first part (Chapter 1 to Chapter 3), we define almost–crystallographic and almost–Bieberbach groups. We spend a lot of time in providing alternative definitions for them. Also we show how the three famous theorems of L. Bieberbach on crystallographic groups can be generalized to the case of almost–crystallographic groups. These first chapters could already suffice to let the reader start his own investigation of almost–crystallographic groups.

2. Chapter 4 forms a part on its own. It deals mainly with my own field of interest, namely the canonical type representations. These are representations of a polycyclic–by–finite group (in our situation always virtually nilpotent), which respect in some sense a given

filtration of that group. We discuss both affine and polynomial representations and present some nice existence and uniqueness results. The reason for considering polycyclic–by–finite groups is natural in the light of Auslander's conjecture.

3. The last part of this monograph (Chapter 5 to Chapter 8) describes a way to classify almost–Bieberbach groups. We also give a complete list of all almost–Bieberbach groups in dimensions ≤ 4, which were obtained using the given method. Moreover, we show how it was possible to use these tables and find in a pure algebraic way some topological invariants (e.g. Betti numbers) of the corresponding infra–nilmanifolds.

Finally, I would like to say a few words of thanks. To Professor Paul Igodt who introduced me to the world of infra–nilmanifolds and who proposed me to investigate the possibility of classifying the almost–Bieberbach groups. I am also grateful to Professor Kyung Bai Lee, since I owe much of my knowledge on almost–Bieberbach groups to him. But most of all I must thank my wife Katleen, for her encouragement when I was doing mathematics in general and especially for her support and practical help when I was writing this book.

Karel Dekimpe,
Kortrijk, August 19, 1996

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