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Finite Geometry and Character Theory

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Preface

It is the aim of this monograph to demonstrate the applications of algebraic number theory and character theory in order to solve problems in finite geometry, in particular problems about difference sets and their corresponding codes and sequences. The idea is the following: Assume that D is an element in the integral group ring $\mathbf{Z}[G]$ which satisfies a certain "basic" equation like $DD^{(-1)} = M$. (We will always assume that G is a finite group.) If G is abelian, we can apply characters χ (or 1-dimensional representations) to compute $\chi(M)$. Using results from algebraic number theory, it is then (sometimes) possible to get informations about $\chi(D)$. This might be helpful to construct D or to prove that no such D can exist which satisfies the basic equation (non-existence). These are the two fundamental problems concerning difference sets, and they will be investigated in this monograph. The classical paper where this approach has been first used is Turyn [167].

In the first chapter we will present the algebraic approach to attack these two problems. In the second chapter we will describe several constructions of difference sets and state some non-existence results. We do not present all known results in complete generality. But the proofs of many theorems about difference sets use ideas which are very closely related to the techniques developed in this monograph, in particular those presented in the first two chapters. I hope that these chapters are particularly useful for those readers who want to become acquainted with "difference set problems".

In most cases our groups are abelian. Of course, one can also ask for solutions of the "basic" equation in $\mathbf{Z}[G]$ if G is non-abelian. Sometimes our techniques are also applicable if G has a "large" homomorphic image which is abelian. But even if this is not the case one can try to use the results from algebraic number theory which are so useful in the abelian case: Instead of 1-dimensional representations one has to consider higher-dimensional representations of the group G. The entries in the matrices of these representations are algebraic integers, which satisfy certain equations (similar to the abelian case), although the equations might be slightly more involved. The reader is referred to the papers by Liebler and Smith [121] and Davis and Smith [56], where this approach has been used quite successfully. In these papers, the same results from algebraic number theory are used which have been already used in the abelian case: Although I restrict myself basically to abelian groups, this book will be also useful for those who are more interested in the non-abelian case.

This monograph is not a book about difference sets in the usual sense (like the books by Baumert [26] and Lander [116]). We describe the known series of difference sets, but this part of the monograph (Section 2.1) has the character of a brief survey. A nice survey about difference sets has been written by Jungnickel [100]. After this survey has been published, several new and exciting results about difference sets (in particular about Hadamard or Menon difference sets and about McFarland difference sets) have been proved (which can be found, without proofs, in this monograph). The only difference sets which we study in more detail are the Singer difference sets and their generalizations in Chapter 3.

In the Chapters 4 and 5 we study several interesting classes of so called relative difference sets. The semiregular case in Chapter 4 is somehow the "relative" analogue of the Hadamard difference sets. In a special case, these semiregular difference sets describe projective planes with a quasiregular collineation group. This is one of three cases which we investigate in Chapter 5. The main problem there is to find numerical restrictions on the possible orders of projective planes admitting quasiregular collineation groups. In one of these cases we can show that n has to be a prime power, or n is a square.

Difference sets and their generalizations give rise to sequences with interesting correlation properties. We investigate these sequences in Chapter 6 (Barker sequences, (almost) perfect sequences). Moreover, we determine the dimensions of certain codes related to difference sets. But we investigate sequences related to difference sets not only in this last chapter. For instance, the Gordon-Mills-Welch difference sets and the corresponding sequences are studied in Chapter 3. Therefore, this monograph can be also useful for mathematicians and engineers who are interested in sequences with "nice" autocorrelation properties: The autocorrelation properties can be translated into group ring equations, and we can use all the techniques developed in this book to find solutions of these equations, or we can try to prove that no solutions can exist. Moreover, we can determine the linear complexities of the sequences.

This monograph is not intended to be a text book: There are too many theorems where we do not give proofs but just refer to the original papers. But this monograph should be a good preparation to read more advanced and technical papers about difference sets. I am sure that the book can be very useful in any course on algebraic design theory (or even algebraic coding theory).

I have posed several questions throughout the text hoping that these will stimulate future research.

Finally, I would like to thank all my colleagues with whom I have worked in the past years and who have supported my research: They all have influenced the presentation in this monograph (without knowing it). In particular, I would like to thank my (former) supervisor Dieter Jungnickel. For the careful reading of (parts of) the final version of the manuscript I thank Bernhard Schmidt and Dieter Jungnickel.

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