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Quantum Probability and Spectral Analysis of Graphs

With a Foreword by Professor Luigi Accardi

With 48 Figures

Springer

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Foreword

It is a great pleasure for me that the new Springer Quantum Probability Programme is opened by the present monograph of Akihito Hora and Nobuaki Obata.

In fact this book epitomizes several distinctive features of contemporary quantum probability: First of all the use of specific quantum probabilistic techniques to bring original and quite non-trivial contributions to problems with an old history and on which a huge literature exists, both independent of quantum probability. Second, but not less important, the ability to create several bridges among different branches of mathematics apparently far from one another such as the theory of orthogonal polynomials and graph theory, Nevanlinna's theory and the theory of representations of the symmetric group.

Moreover, the main topic of the present monograph, the asymptotic behaviour of large graphs, is acquiring a growing importance in a multiplicity of applications to several different fields, from solid state physics to complex networks, from biology to telecommunications and operation research, to combinatorial optimization. This creates a potential audience for the present book which goes far beyond the mathematicians and includes physicists, engineers of several different branches, as well as biologists and economists.

From the mathematical point of view, the use of sophisticated analytical tools to draw conclusions on discrete structures, such as, graphs, is particularly appealing. The use of analysis, the science of the continuum, to discover non-trivial properties of discrete structures has an established tradition in number theory, but in graph theory it constitutes a relatively recent trend and there are few doubts that this trend will expand to an extent comparable to what we find in the theory of numbers.

Two main ideas of quantum probability form the unifying framework of the present book:

1. The quantum decomposition of a classical random variable.
2. The existence of a multiplicity of notions of quantum stochastic independence.

The authors establish original and fruitful connections between these ideas and graph theory by considering the adjacency matrix of a graph as a classical random variable and then by decomposing it in two different ways:

(i) either using its quantum decomposition;
(ii) or decomposing it into a sum of independent quantum random variables (for some notion of quantum independence).

The former method has a universal applicability but depends on the choice of a *stratification* of the given graph. The latter is applicable only to special types of graphs (those which can be obtained from other graphs by applying some notion of product) but does not depend on special choices.

In both cases these decompositions allow to reduce many problems related to the asymptotics of large graphs to traditional probabilistic problems such as quantum laws of large numbers, quantum central limit theorems, etc. Given the central role of these two decompositions in the present volume, it is maybe useful for the reader to add some intuitive and qualitative information about them.

The quantum decomposition of a classical random variable, like many other important mathematical ideas, has a long history. Its first examples, the representation of the Gaussian and Poisson measures on $\mathbb{R}^d$ in terms of creation and annihilation operators, were routinely used in various fields of quantum theory, in particular quantum optics. Its continuous extension, obtained by the usual second quantization functor, played a fundamental role in Hudson–Parthasarathy quantum stochastic calculus and a few additional examples, going beyond the Gaussian and Poisson family appeared in the early 1990s in papers by Bożejko and Speicher.

However, the realization that the quantum decomposition of a classical random variable is a universal phenomenon in the category of random variables with moments of all orders came up only in connection with the development of the theory of interacting Fock spaces. This theory provided the natural conceptual framework to interpret the famous Jacobi relation for orthogonal polynomials in terms of a new class of creation, annihilation and preservation operators generalizing in a natural way the corresponding objects in quantum mechanics.

Most of the present monograph deals with the quantum decomposition of a single real valued random variable for which the quantum decomposition is just a re-interpretation of the Jacobi relation. The situation radically changes for $\mathbb{R}^d$-valued random variables with $d \geq 2$ for which a natural (i.e. intrinsic) extension of the Jacobi relation could only be formulated in terms of interacting Fock space.

An interesting discovery of the authors of the present book is that examples of this more complex situation also arise in connections with graph theory. This will be surely a direction of further developments for the theory developed in the present monograph.

The intimately related notions of quantum decomposition of a classical random variable and of interacting Fock space have been up to now two of the most fruitful and far reaching new ideas introduced by quantum probability. The authors of the present monograph have developed in the past years a new approach to a traditional problem of mathematics, the asymptotics of large graphs, which puts to use in an original and creative way both the above-mentioned notions.

The results of their efforts enjoy the typical merits of inspiring mathematics: elegance and depth. In fact a vast multiplicity of results, previously obtained at the cost of lengthy and ad hoc calculations or complicated combinatorial arguments, are now obtained through a unified method based on the common intuition that the quantum decomposition of the adjacency matrix of the limit graph should be the limit of the quantum decompositions of the adjacency matrices of the approximating graphs. This limit procedure involves central limit theorems which, in the previous approaches to the asymptotics of large graphs, were proved within the context of classical probability. In the present monograph they are proved in their full quantum form and not just in their reduced classical (or semiclassical) form. This produces the usual advantage of quantum central limit theorems with respect to classical ones namely that, by considering various types of self-adjoint linear combinations of the quantum random variables, one obtains the corresponding central limit theorem for the resulting classical random variable.

Thus in some sense a quantum central limit theorem is equivalent to infinitely many classical central limit theorems. This additional degree of freedom was little appreciated in the early quantum central limit theorems, concerning Boson, Fermion, q-deformed, free random variables, because, before the discovery of the universality of the quantum decomposition of classical random variables, a change in the coefficients of the linear combination, could imply a radical change (i.e., not limited to a simple change of parameters within the same family) in the limit classical distribution, only at some critical values of the parameters (e.g., if a^+, a^- are Boson Fock random variables, then independently of z the Boson Fock vacuum distribution of $za^+ + \bar{z}a^- + \lambda a^+a^-$ is Gaussian for $\lambda = 0$ and Poisson for $\lambda \neq 0$).

The emergence of the interacting Fock space produced the first examples (due to Lu) in which a continuous interpolation between radically different measures could occur by continuous variations of the coefficients of the linear combinations of a^+ and a^-. This bring us to the second deep and totally unexpected connection between quantum probability and graphs, which is investigated in the present monograph starting from Chap. 8. To explain this idea let us recall that one of the basic tenets of quantum probability since its development in the early 1970s has been the multiplicity of notions of independence. The first examples beyond classical independence (Bose and Fermi independence) where motivated by physics and the first notions of independence going beyond these physically motivated ones were introduced by von

Waldenfels in the early 1970s. However, it is only with the birth of free probability, in the late 1980s, that the notions of stochastic independence begin to proliferate and to motivate theoretical investigations trying to unify them within some common framework.

An important step in this direction, because of its constructive and not merely descriptive nature, was Lenczewski's tensor representation of the Boolean m-free and free independence, extended to the monotone case by Franz and Muraki (this extension was also implicitly used in an earlier paper by Liebscher). This tensor representation turned out to be absolutely crucial in the connection between notions of independence and graphs, which can be described by the following general abstract ansatz: 'there exist many different notions of products among graphs and, if π is such a notion, the adjacency matrix of a π-product of two graphs can be decomposed as a non-trivial sum of $\mathcal{I}_\pi$-independent quantum random variables where $\mathcal{I}_\pi$ denotes a notion of independence determined by the product π and by a vector in the l^2-space of the graph'. It is then natural to call this decomposition the π-decomposition of the adjacency matrix of the product graph.

Comparing this with a folklore ansatz of quantum probability, namely: 'to every notion of π-product among algebras, one can associate a notion $\mathcal{I}_\pi$ of stochastic independence' one understands that the analogy between the two statements is a natural fact because, by exploiting the equivalence (of categories) between sets and complex valued functions on them, one can always translate a notion of product of graphs into a notion of product of algebras and conversely.

Historically, the first example which motivated the above-mentioned ansatz was the discovery that the adjacency matrix of a comb product of a graph with a rooted graph can be decomposed as the sum of two monotone independent random variables (with respect to a natural product vector). In other words: the above ansatz is true if π is the comb product among graphs and $\mathcal{I}_\pi$ the notion of monotone independence. In addition the π-decomposition of the adjacency matrix is nothing but a particular realization of the tensor representation of two monotone independent random variables.

As expected, if π is the usual cartesian product the corresponding independence notion $\mathcal{I}_\pi$ is the usual tensor (or classical) independence. The fact that, if π is the star-product of rooted graphs, then the associated notion of independence $\mathcal{I}_\pi$ is Boolean independence was realized in a short time by a number of people. Strangely enough the fourth notion of independence in Schürman's axiomatization, i.e. free independence, was the hardest one to relate to a product of graphs in the sense of the above ansatz. This is strange because the free product of graphs was introduced by Znoǐko about 30 years ago and then studied by many authors, in particular Gutkin and Quenell, thus it would have been natural to conjecture that the free product of graphs should be related to free independence.

That this is true has been realized only recently, but the relation is not as simple as in the case of the previous three independences. In fact, in the formerly known cases, the adjacency matrix of the π-product of two graphs was decomposed into a sum of two $\mathcal{I}_\pi$-independent quantum random variables, but in the free case the π-decomposition involves infinitely many free independent random variables. Another special feature of the free product is that it can be expressed by 'combining together' (in some technical sense) the comb (monotone) and the star (Boolean) products.

These arguments are not dealt with in the present book because fortunately the authors realized that, if one decides to include all the important latest developments in a field evolving at the pace of quantum probability, then the present monograph would have become a Godot.

Another important quality of the present volume is the authors' ability to condensate a remarkably large amount of information in a clear and self–contained way. In the structure of this book one can clearly distinguish three parts, approximatively of the same length (about 100 pages). The first part introduces all the basic notions of quantum probability, analysis and graph theory used in the following. The second part (from Chaps. 4 to 8) deals with different types of graphs and the last part (from Chaps. 9 to 12) includes an introduction to Kerov's theory of the asymptotics of the representations of the permutation group $S(N)$, for large N, and the extensions of this theory in various directions, due to various authors themselves and other researchers.

The clarity of exposition, the ability to keep the route firmly aimed towards the essential issues, without digressions on inessential details, the wealth of information and the abundance of new results make the present monograph a precious reference as well as an intriguing source of inspiration for all those who are interested in the asymptotics of large graphs as well as in any of the multiple applications of this theory.

Roma
December, 2006 *Luigi Accardi*

Preface

Quantum probability theory provides a framework of extending the measure-theoretical (Kolmogorovian) probability theory. The idea traces back to von Neumann [219], who, aiming at the mathematical foundation for the statistical questions in quantum mechanics, initiated a parallel theory by making a self-adjoint operator and a trace play the roles of a random variable and a probability measure, respectively. During the recent development, quantum probability theory has been related to various fields of mathematical sciences beyond the original purposes. We focus in this book on the spectral analysis of a large graph (or of a growing graph) and show how the quantum probabilistic techniques are applied, especially, for the study of asymptotics of spectral distributions in terms of quantum central limit theorem.

Let us explain our basic idea with the simplest example. The coin-toss is modelled by a Bernoulli random variable X specified by

$$P(X = +1) = P(X = -1) = \frac{1}{2}, \tag{0.1}$$

or more essentially by its distribution, i.e., the probability measure μ defined by

$$\mu = \frac{1}{2}\,\delta_{-1} + \frac{1}{2}\,\delta_{+1}. \tag{0.2}$$

The moment sequence is one of the most fundamental characteristics of a probability measure. For μ in (0.2) the moment sequence is calculated with no difficulty as

$$M_m(\mu) = \int_{-\infty}^{+\infty} x^m \mu(dx) = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \tag{0.3}$$

When we wish to recover a probability measure from the moment sequence, we meet in general a delicate problem called *determinate moment problem.* For the coin-toss there is no such an obstacle and we can recover the Bernoulli distribution from the moment sequence.

Now we discuss, somehow abruptly, elementary linear algebra. We set

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{0.4}$$

Then $\{e_0, e_1\}$ is an orthonormal basis of the two-dimensional Hilbert space $\mathbb{C}^2$ and A is a self-adjoint operator acting on it. It is straightforward to see that

$$\langle e_0, A^m e_0 \rangle = \begin{cases} 1, & \text{if } m \text{ is even}, \\ 0, & \text{otherwise}, \end{cases} \tag{0.5}$$

which coincides with (0.3). In other words, the coin-toss is also modelled by using the two-dimensional Hilbert space $\mathbb{C}^2$ and the matrix A. In our terminology, letting $\mathcal{A}$ be the $*$-algebra generated by A, the coin-toss is modelled by an algebraic random variable A in an algebraic probability space $(\mathcal{A}, e_0)$. We call A an *algebraic realization* of the random variable X.

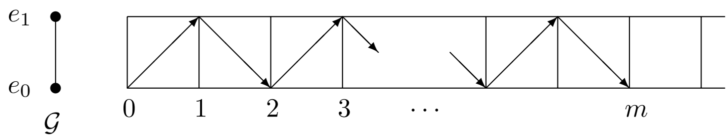
Once we come to an algebraic realization of a classical random variable, we are naturally led to the non-commutative paradigm. Let us consider the decomposition

$$A = A^+ + A^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{0.6}$$

which yields a simple proof of (0.5). In fact, note first that

$$\langle e_0, A^m e_0 \rangle = \langle e_0, (A^+ + A^-)^m e_0 \rangle = \sum_{\epsilon_1, \dots, \epsilon_m \in \{\pm\}} \langle e_0, A^{\epsilon_m} \cdots A^{\epsilon_1} e_0 \rangle. \tag{0.7}$$

Let $\mathcal{G}$ be a connected graph consisting of two vertices e_0, e_1. Observing the obvious fact that (0.7) coincides with the number of m-step walks starting at and terminating at e_0 (see the figure below), we obtain (0.5).



Thus, the computation of the mth moment of A is reduced to counting the number of certain walks in a graph through (0.6). This decomposition is in some sense canonical and is called the *quantum decomposition* of A.

We now note that A in (0.4) is the adjacency matrix of the graph $\mathcal{G}$. Having established the identity

$$\langle e_0, A^m e_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots, \tag{0.8}$$

we say that μ is the spectral distribution of A in the state e_0. In other words, we obtain an integral expression for the number of returning walks in the graph by means of such a spectral distribution. A key role in deriving (0.8) is again played by the quantum decomposition.

The *method of quantum decomposition* is the central topic of this book. Given a classical random variable, or a probability distribution, we consider the associated orthogonal polynomials. We then introduce the quantum decomposition through the famous three-term recurrence relation and come to the fundamental link with an interacting Fock probability space, which is one of the most basic algebraic probability space. On this basis we shall develop spectral analysis of a graph by regarding the adjacency matrix as an algebraic random variable and illustrate with many concrete examples usefulness of the method of quantum decomposition. Our method is effective especially for the asymptotic spectral analysis and the results are formulated in terms of *quantum central limit theorems*, where our target is not a single graph but a growing graph. Making a sharp contrast with the so-called harmonic analysis on discrete structures, our approach shares a common spirit with the asymptotic combinatorics proposed by Vershik and is expected to contribute also the interdisciplinary study of evolution of networks. Spectral analysis of large graphs is an interesting field in itself, which has a wide range of communications with other disciplines. At the same time it enables us to see pleasant aspects in which quantum probability essentially meets profound classical analysis.

This book is organized as follows: Chapter 1 is devoted to assembling basic notions and notations in quantum probability theory. A special emphasis is placed on the interplay between interacting Fock probability spaces and orthogonal polynomials. The Stieltjes transform and its continued fraction expansion is concisely and self-containedly reviewed.

Chapter 2 gives a short introduction to graph theory and explains our main questions. The idea of quantum decomposition is applied to the adjacency matrix of a graph.

Chapter 3 deals with distance-regular graphs which possess a significant property from the viewpoint of quantum decomposition. We shall establish general framework for asymptotic spectral distributions of the adjacency matrix and derive the limit distributions in terms of intersection numbers.

Chapter 4 analyses homogeneous trees as the first concrete example of growing distance-regular graphs. We shall derive the Wigner semicircle law from the vacuum state and the free Poisson distribution from the deformed vacuum state. The former is a reproduction of the free central limit theorem.

Chapter 5 studies the Hamming graphs which form a growing distance-regular graph. Both Gaussian and Poisson distributions emerge as the central limit distributions.

Chapter 6 discusses the Johnson graphs and odd graphs as further examples of growing distance-regular graphs. As the central limit distributions, we shall obtain the exponential distribution and the geometric distribution from the Johnson graphs, and the two-sided Rayleigh distribution from the odd graphs.

Chapter 7 focuses on growing regular graphs. We shall prove the central limit theorem under some natural conditions, which cover many concrete examples.

Chapter 8 surveys four basic notions of independence in quantum probability theory. The adjacency matrix of an integer lattice is decomposed into a sum of commutative independent random variables, which is also observed through Fourier transform. While, the adjacency matrix of a homogeneous tree is decomposed into a sum of free independent random variables, which provide a prototype of free central limit theorem of Voiculescu. For the rest notions of independence, i.e., the Boolean independence and the monotone independence, we assign a particular graph structure called *star product* and *comb product* and study asymptotic spectral distributions as an application of the associated central limit theorems.

Chapter 9 is devoted to assembling basic notions and tools in representation theory of the symmetric groups. The analytic description of Young diagrams, which is essential for the study of asymptotic behaviour of a representation of $S(n)$ as $n \to \infty$, is also concisely overviewed.

Chapter 10 attempts to derive the celebrated limit shape of Young diagrams, which opens the gateway to the asymptotic representation theory of the symmetric groups. Our approach is based on the moment method developed in previous chapters and serves as a new accessible introduction to asymptotic representation theory.

Chapter 11 answers to the natural question about the fluctuation in a small neighbourhood of the limit shape of Young diagrams with respect to the Plancherel measure. The nature of Gaussian fluctuation is described from several points of view, especially as central limit theorem for quantum components of adjacency matrices associated with conjugacy classes.

Finally Chap. 12 studies a one-parameter deformation (called α-deformation) related to the Jack measure on Young diagrams and the Metropolis algorithm on the symmetric group. The associated central limit theorem follows from the quantum central limit theorem (Theorem 11.13), which shows again usefulness of quantum decomposition.

The notes section at the end of each chapter contains supplementary information of references but is not aimed at documentation. Accordingly, the bibliography contains mainly references that we have actually used while writing this book, and therefore, is far from being complete.

We are indebted to many people whose books, papers and lectures inspired our approach and improved our knowledge, especially, K. Aomoto, M. Bożejko, F. Hiai and D. Petz. Special thanks are due to L. Accardi for stimulating discussion, constant encouragement and kind invitation of writing this book.

Okayama and Sendai
January, 2006

Akihito Hora
Nobuaki Obata

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