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On Artin's Conjecture for Odd 2-dimensional Representations

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Foreword

The investigation of the relation between the arithmetic of Galois representations and the analytic behaviour of corresponding L-series is one of the central topics in current number theory and arithmetic geometry.

One of the outstanding problems in this field of research is Artin's conjecture which predicts the holomorphy of the Artin L-series of non-trivial irreducible complex representations of the absolute Galois group of number fields. Till today this conjecture is not known to be true in general; in fact even in the case of two-dimensional representations one has only partial results due to work of Hecke, Langlands and Tunnell. From this it follows that if φ is an irreducible odd two-dimensional representation of the Galois group of $\mathbb{Q}$ then Artin's conjecture is true if the projective image of φ is either a dihedral group (Hecke) or A_4 (Langlands) or S_4 (Tunnell). The only remaining possibility is that this image is equal to A_5, and in this case the answer to Artin's conjecture is not known.

The main concern of the following work is to develop and to implement algorithms by which one can verify Artin's conjecture for odd two-dimensional representations in a fairly wide range; it relies on considerations of J. Buhler contained in his book: "Icosahedral Galois Extensions" where he solves the problem for representations with Artin conductor 800, and in fact the attempt to systematize his work was one of the motivations of the project.

Beyond this we think that the results obtained and especially the algorithms that allow us to compute spaces of cusp forms efficiently will be of use for many other problems related to arithmetical questions involving modular forms. We only want to mention here that L. Merel recently was able to use his results on the action of Hecke operators on Manin symbols to prove the existence of a bound for the order of the torsion group of elliptic curves over number fields K depending only on the degree of K.

The different chapters of the book are written by several authors involved in the project and so they are independent papers in some sense. Nevertheless we hope to convince the reader that the book is not just a loose collection but that it reflects the joint attempt of the members of the team to understand the arithmetic of two-dimensional Galois representations of the Galois group of $\mathbb{Q}$ better.

Our starting point were the beautiful results of Deligne-Serre and Langlands-Weil that there is a one-to-one correspondence between new forms of weight one of level N and Nebentype ϵ and representations as above with Artin conductor N and determinant ϵ satisfying Artin's conjecture. So to verify this conjecture one has to show that the number of irreducible odd two-dimensional representations of given conductors and determinant is equal to the dimension of the corresponding space of new forms.

To determine representations of D_n-type one can use class field theory, for A_4-resp. S_4-type representation there is a nice geometric interpretation using elements of order 2 in Selmer groups of elliptic curves which can be transformed into an effective algorithm. If the image of the projective representation is equal to A_5 the situation is more difficult. For practical purposes we computed a table of all A_5-extensions with determinant $\leq 2083^2$ and used this table to determine the corresponding representations. Results of Klein and Serre are applied to construct such extensions "geometrically" with the help of 5-torsion points of elliptic curves and of Jacobians of genus 2 but until now it seems that this approach does not lead to an implementation of an effective algorithm.

The second task is the computation of the dimensions of spaces of cusp forms of weight 1. To do this we compute spaces of cusp forms of weight ≤ 2 by determining "enough" Fourier coefficients of a $\mathbb{Z}$-base and then we identify subspaces isomorphic to spaces of forms of weight one by solving linear equations satisfied by these Fourier coefficients. An essential tool to do this is provided by Manin symbols. Since L. Merel and X. Wang independently developed methods for these computations that eventually merged into a very satisfying algorithm we decided to give two expositions written by Merel resp. Wang with a certain amount of overlap in which the background of this algorithm is described. For the practical implementation of the algorithm the assistance of W. Happle was of great importance.

For technical reasons and to make some discussions simpler we restricted ourselves to the case where the determinant is a quadratic character. To keep the amount of computation manageable we assume that $N = 2^\alpha \cdot n$ with n odd and square free and $lcm(N, 4N) \leq 10000$, and it is no surprise that Artin's conjecture is proved to be true in all cases. In fact A_5-type representations occur for seven conductors in this range with determinant equal to χ_{-n}. In these examples we have: $N = 4n$, $n \in \{487, 751, 887, 919, 2083\}$ resp. $N = 2^5 n$, $n \in \{73, 193\}$.

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