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Shadowing in Dynamical Systems

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To my sons Sergei and Kirill

Preface

Let (X, r) be a metric space and let ϕ be a homeomorphism mapping X onto itself.

A d-pseudotrajectory of the dynamical system ϕ is a sequence of points $\xi = \{x_k \in X : k \in \mathbb{Z}\}$ or $\xi = \{x_k \in X : k \in \mathbb{Z}_+\}$ such that

$$r(\phi(x_k), x_{k+1}) < d.$$

Usually, a pseudotrajectory is considered as a result of application of a numerical method to our dynamical system ϕ. In this case, the value d measures one-step errors of the method and round-off errors.

The notion of a pseudotrajectory plays an important role in the general qualitative theory of dynamical systems. It is used to define some types of invariant sets (such as the chain-recurrent set [Con] or chain prolongations [Pi2]).

We say that a point x (ϵ, ϕ)-shadows a pseudotrajectory $\xi = \{x_k\}$ if the inequalities

$$r(\phi^k(x), x_k) < \epsilon$$

hold. Thus, the existence of a shadowing point for a pseudotrajectory ξ means that ξ is close to a real trajectory of ϕ.

The mostly studied shadowing property of dynamical systems is the POTP (the pseudoorbit tracing property). A system ϕ is said to have the POTP if given $\epsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory ξ there is a point x that (ϵ, ϕ)-shadows ξ.

From the numerical point of view, if ϕ has the POTP, then numerically obtained trajectories (on arbitrarily long time intervals) reflect the real behavior of trajectories of ϕ.

If ψ is a dynamical system C^0-close to ϕ, then obviously any trajectory of ψ is a d-pseudotrajectory of ϕ with small d. Thus, if ϕ has the POTP, then any trajectory of the "perturbed" system ψ is close to a trajectory of ϕ. Hence, we may consider the POTP as a weak form of stability of ϕ with respect to C^0-small perturbations.

Theory of shadowing was developed intensively in recent years and became a significant part of the qualitative theory of dynamical systems containing a lot of interesting and deep results. This book is an introduction to the main methods of shadowing.

The book is addressed to the following three main groups of readers.

The main expected group of readers are specialists in the qualitative theory of dynamical systems and its applications. For them, the author tried to describe a unified approach based on shadowing results for sequences of mappings of Banach spaces. It is shown that this approach can be applied to establish the classical shadowing property and limit shadowing properties in a neighborhood of a hyperbolic set, shadowing properties of structurally stable dynamical systems (both diffeomorphisms and flows), and some other classes of shadowing properties. In addition, we present a systematic treatment of connections between the shadowing theory and the classical fields of the global qualitative theory of dynamical systems (such as the theories of topological stability and of structural stability).

Next, some parts of the book (Sects. 1.1, 1.2.1, 1.2.2, 1.2.3, 1.3.1, 1.3.2, 3.1, 3.2, and 4.1) can be included into courses or used for the first acquaintance with the theory of shadowing by advanced students with basic training in dynamical systems. For this purpose, main definitions and results are illustrated by a lot of simple (maybe, too simple for specialists) examples.

Proofs of basic results of the theory and description of some important general constructions contained in the sections mentioned above are given with all details and with necessary background from functional analysis.

Finally, the book is addressed to specialists in numerical methods for dynamical systems. Some recent conferences (for example, the Conference on Dynamical Numerical Analysis, Georgia Tech, December 1995) showed that the idea of shadowing plays now an important role in this field and that "numerical dynamics" specialists need a detailed survey of results and methods of the shadowing theory.

It was an intention of the author to describe two "numerically oriented" shadowing approaches. The first one is based on methods for verification of numerically obtained data. These methods allow to establish the existence of a real trajectory near a computed one and to give the corresponding error bounds (see [Cho2, Cho3, Coo1-Coo6, Gr, Ham, Sau2] and others). The second approach establishes shadowing properties of dynamical systems generated by numerical methods (for example, discretizations of a parabolic PDE are realized as finite-dimensional diffeomorphisms [Ei1], see Sect. 4.4). These results allow us to study the influence of errors in application of numerical methods on unbounded time intervals.

The book consists of 4 chapters. Chapter 1 is devoted to "local shadowing", i.e., shadowing in a neighborhood of an invariant set. We introduce the main shadowing properties in Sect. 1.1 and discuss relations between these properties.

Section 1.2 is devoted to the classical shadowing result – the Shadowing Lemma by Anosov [Ano2] and Bowen [Bo2]. This result states that a diffeomorphism has the POTP in a neighborhood of its hyperbolic set. It is shown that this shadowing property is Lipschitz, i.e., if Λ is a hyperbolic set of a diffeomorphism ϕ, then there exist constants $d_0, L > 0$ and a neighborhood U of Λ such that for any sequence $\{x_k\} \subset U$ with

$$r(\phi(x_k), x_{k+1}) \leq d \leq d_0 \tag{0.1}$$

there exists a point x with the property

$$r(\phi^k(x), x_k) \leq Ld. \tag{0.2}$$

In Subsect. 1.2.1, we describe main properties of hyperbolic sets. It is shown that there exists a Lyapunov metric in a neighborhood of a hyperbolic set. In Subsect. 1.2.2, we give a detailed proof of the Shadowing Lemma applying the approach of Anosov [Ano2].

Let Λ be a hyperbolic set of a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$. Assume that, for a sequence $\{x_k\}$ belonging to a small neighborhood of Λ, inequalities (0.1) hold. Following Anosov, we study a functional equation to find a sequence $\{v_k \in \mathbb{R}^n\}$ such that

$$\phi(x_k + v_k) = x_{k+1} + v_{k+1} \tag{0.3}$$

(i.e., the sequence $\{x_k + v_k\}$ is a trajectory of ϕ), and the point $x = x_0 + v_0$ satisfies inequalities (0.2).

Subsection 1.2.3 is devoted to the so-called "theorem on a family of ϵ-trajectories" [Ano2] and some of its applications. This statement says that if Λ is a hyperbolic set for a diffemorphism ϕ, and if a neighborhood of Λ contains a family of approximate trajectories of a diffeomorphism ψ, C^1-close to ϕ, then it is possible to shadow all this family by trajectories of ψ.

In Subsect. 1.2.4, Bowen's proof of the Shadowing Lemma is given. We also describe an abstract construction modelling the method of Bowen (the so-called Smale space) introduced by Ruelle [Ru].

In Sect. 1.3, we introduce the main technical tool applied in this book – "abstract" shadowing results for sequences of mappings of Banach spaces [Pi4]. In the case of a hyperbolic set of a diffeomorphism ϕ, we can introduce "local" mappings

$$\phi_k(v) = \phi(x_k + v) - x_{k+1},$$

then Eqs. (0.3) have the form

$$\phi_k(v_k) = v_{k+1}. \tag{0.4}$$

Generalizing this approach, we consider sequences $\{H_k\}$ of Banach spaces and mappings $\phi_k : H_k \to H_k$, and look for sequences $\{v_k \in H_k\}$ satisfying (0.4). Obviously, inequalities (0.1) are equivalent to the estimates

$$|\phi_k(0)| \le d \le d_0, \tag{0.5}$$

and inequalities (0.2) are equivalent to the estimates

$$|v_k| \le Ld. \tag{0.6}$$

In Subsect. 1.3.1, we study sequences $\{\phi_k\}$ with (possibly) nonivertible linear parts. Theorem 1.3.1 (Subsect. 1.3.1) gives sufficient conditions under which there exist constants d_0 and L such that inequalities (0.5) imply the existence of a sequence $\{v_k\}$ satisfying (0.6). Such a sequence is considered as a local shadowing trajectory for the sequence $\{\phi_k\}$.

In Subsect. 1.3.2, we obtain conditions for the uniqueness of a shadowing trajectory in this case. Subsection 1.3.3 contains a general scheme of application of Theorem 1.3.1. This scheme is applied to establish an analog of the Shadowing Lemma for one-sided sequences. Subsection 1.3.4 is devoted to two theorems on shadowing for sequences of mappings of Banach spaces proved by Chow, Lin, and Palmer [Cho1] and by Steinlein and Walther [Ste1].

Pliss [Pli2, Pli3] obtained necessary and sufficient conditions under which a family of systems of linear differential equations has uniformly bounded solutions. We "translate" these results into the shadowing language in Subsect. 1.3.5 and apply them to "affine" mappings

$$\phi_k(v) = A_k v + w_{k+1},\ v \in \mathbb{R}^n.$$

It is shown that the developed method can be also applied to nonlinear mappings ϕ_k (Theorem 1.3.7).

In the statement of the shadowing problem above, the values

$$d_k = r(x_{k+1}, \phi(x_k))$$

are assumed to be uniformly small. One can impose another condition on these values,

$$d_k \to 0 \text{ as } k \to \infty,$$

and look for a point x such that the values

$$h_k = r(\phi^k(x), x_k) \to 0 \text{ as } k \to \infty.$$

The corresponding shadowing property (called the limit shadowing property [Ei2]) is studied in Sect. 1.4. It is shown in Subsect. 1.4.1 that in a neighborhood of its hyperbolic set a diffeomorphism has this property.

In Subsect. 1.4.2, we investigate the rate of convergence of the values h_k in terms of d_k. It is shown that if a sequence $\{x_k : k \geq 0\}$ belongs to a neighborhood U of a hyperbolic set of a diffeomorphism ϕ, and, for some $p \geq 1$, the $\mathcal{L}_p$ norm

$$||\{d_k\}||_p = \left(\sum_{k\geq 0} d_k^p\right)^{1/p}$$

is small, then there is a point x such that

$$||\{h_k\}||_p \leq L||\{d_k\}||_p$$

with a constant L depending only on the neighborhood U.

In Subsect. 1.4.3, we pass from the spaces $\mathcal{L}_p$ to their weighted analogs, the spaces $\mathcal{L}_{\bar{r},p}$ with norms

$$||\{d_k\}||_{\bar{r},p} = \left(\sum_{k\geq 0} r_k d_k^p\right)^{1/p},\ \bar{r} = \{r_k\}.$$

We show that it is possible to establish the corresponding "$\mathcal{L}_{\bar{r},p}$-shadowing property" in a neighborhood of a compact invariant set, not necessarily hyperbolic, under appropriate conditions on the weight sequence $\bar{r}$. These conditions are formulated in terms of the so-called Sacker-Sell spectrum [Sac].

Another possibility to establish the "$\mathcal{L}_{\bar{r},p}$"-shadowing is to assume that the weight sequence $\bar{r}$ grows "fast enough" (Theorems 1.4.6 and 1.4.8).

Hirsch studied in [Hirs4] asymptotic pseudotrajectories, i.e., sequences $\{x_k\}$ such that

$$\overline{\lim}_{k\to\infty} d_k^{1/k} \le \lambda, \text{ where } \lambda \in (0,1).$$

The main result of Hirsch [Hirs4] on shadowing of asymptotic pseudotrajectories is described in Subsect. 1.4.4.

Section 1.5 is devoted to shadowing in flows generated by autonomous systems of ordinary differential equations. Technically, the shadowing problem for flows is significantly more complicated than the one for discrete dynamical systems.

We prove in this section that if Λ is a hyperbolic set of a flow containing no rest points, then in a neighborhood of Λ pseudotrajectories are shadowed by real trajectories, and the shadowing is Lipschitz with respect to the "errors".

In Chap. 2, we study connections between shadowing properties and the classical "global" properties of dynamical systems, such as topological stability and structural stability. We consider dynamical systems on a closed smooth manifold.

Walters [Wa2] and Morimoto [Moriml] showed that a topologically stable homeomorphism has the POTP. We prove this statement in Sect. 2.1. It is also shown in this section that an expansive homeomorphism having the POTP is topologically stable.

Section 2.2, the main part of Chap. 2, is devoted to shadowing properties of structurally stable dynamical systems. In Subsect. 2.2.1, we prove that a structurally stable flow has a Lipschitz shadowing property [Pi4]. We begin to work with a flow since this case is technically more difficult than the case of a diffeomorphism (mostly due to the possible coexistence of rest points and of nonwandering trajectories that are not rest points). The main statement (Theorem 2.2.3) is reduced to shadowing results for sequences of mappings of Banach spaces with noninvertible "linear parts" (see Sect. 1.3). It was an intention of the author to make the presentation of Theorem 2.2.3 maximally "self-contained". Due to this reason, we give a detailed proof of the existence of Robinson's "compatible extensions of stable and unstable bundles" [Robil] (see Lemma 2.2.9).

Shadowing for structurally stable diffeomorphisms is studied in Subsect. 2.2.2. It is shown that a structurally stable diffeomorphism has the Lipschitz shadowing property. Sakai noted that the POTP is "uniform" in a C^1-neighborhood of a structurally stable diffeomorphism [Sakl] and that the C^1 interior of the set of diffeomorphisms with the POTP consists of structurally stable diffeomorphisms [Sak2]. We prove that the Lipschitz shadowing property is also uniform [Beg] and that a diffeomorphism in the C^1 interior of diffeomorphisms with this property is structurally stable.

Without additional assumptions on the dynamical system, no necessary conditions for the POTP are known. In Sect. 2.3, we study necessary and sufficient conditions of shadowing for two-dimensional diffeomorphisms satisfying Axiom A. We show that in this case the POTP is equivalent to the so-called C^0

transversality condition [Sak3], and the Lipschitz shadowing property is equivalent to the strong transversality condition (and hence to structural stability).

In the same section, we describe results of Plamenevskaya [Pla2] on weak shadowing for two-dimensional Axiom A diffeomorphisms. A diffeomorphism ϕ is said to have the weak shadowing property if given $\epsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory ξ there is a trajectory $O(x)$ of ϕ with the property

$$\xi \subset N_\epsilon(O(x))$$

(here $N_\epsilon(O(x))$ is the ϵ-neighborhood of $O(x)$). An example of an Axiom A diffeomorphism of the two-torus T^2 with finite nonwandering set shows that necessary and sufficient conditions for weak shadowing have complicated structure, they are connected with arithmetical properties of eigenvalues of the derivative $D\phi$ at saddle fixed points.

Section 2.4 is devoted to the long-standing problem of genericity of the shadowing property when the dimension of the manifold is arbitrary. We formulate (without a proof) a theorem from [Pi5] stating that a C^0-generic homeomorphism has the POTP.

In Chap. 3, we study the shadowing problem for some special classes of dynamical systems. Section 3.1 is devoted to one-dimensional systems. We prove two theorems of Plamenevskaya [Pla1]. The first one gives necessary and sufficient conditions under which a homeomorphism of the circle has the POTP. In the second theorem, sufficient conditions for the limit shadowing property are obtained. As a corollary, it is shown that if a homeomorphism of the circle has the POTP, then it has the limit shadowing property.

In Sect. 3.2, we consider linear and linearly induced systems. Following Morimoto [Morim3], we show that a linear mapping $\phi(x) = Ax$ has the POTP if and only if the matrix A is hyperbolic. Conditions of the POTP are known for a wide class of linearly induced systems, we treat in detail the so-called spherical linear transformations of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ defined by the formula

$$\phi(x) = \frac{Ax}{|Ax|}.$$

We prove the following theorem of Sasaki [Sas]: ϕ has the POTP if and only if the eigenvalues of the matrix A have different absolute values.

The second part of Chap. 3 is devoted to two special classes of infinite-dimensional dynamical systems. In both cases, the shadowing problem is reduced to the same problem for auxiliary finite-dimensional systems.

The first class, lattice systems, is studied in Sect. 3.3. Usually, the following three types of solutions for lattice systems are investigated: steady-state solutions, travelling waves, and spatially-homogeneous solutions. Under appropriate conditions, these solutions are governed by finite-dimensional diffeomorphisms. We describe conditions [Af3] under which an approximately static (approximately travelling or approximately spatially-homogeneous) pseudotrajectory is shadowed by a steady-state solution (correspondingly, by a travelling wave or by a spatially-homogeneous solution).

Section 3.4 is devoted to shadowing in nonlinear evolution systems on Hilbert spaces. It is assumed that the evolution system $\mathcal{S}$ generated by a parabolic PDE

$$u_t = u_{xx} + f(u) \tag{0.7}$$

has Morse-Smale structure on its global attractor $\mathcal{A}$. We show that $\mathcal{S}$ has a type of Lipschitz shadowing property in a neighborhood of the global attractor $\mathcal{A}$ [Lar4].

In the last chapter, we describe some applications of the shadowing theory to numerical investigation of dynamical systems. Section 4.1 is devoted to methods of verification of numerically obtained data. We prove two theorems of Chow and Palmer [Cho1, Cho2] on finite shadowing in one-dimensional and multidimensional systems.

Coomes, Koçak, and Palmer [Coo1-Coo6] developed a theory of "practical" shadowing for ordinary differential equations. Their methods allow to establish the existence of a real trajectory near a computed one. Section 4.2 is devoted to one of their methods, the method of periodic shadowing [Coo2]. This method gives computable error bounds for the distance between a computed closed trajectory and a real one.

In Sect. 4.3, we consider connections between pseudotrajectories of a dynamical system and its "spectral" characteristics such as Lyapunov exponents and the Morse spectrum. In Subsect. 4.3.1, we study the problem of approximate evaluation of Lyapunov exponents. It is shown that in the evaluation of the upper Lyapunov exponent on a hyperbolic set, the resulting errors are Lipschitz with respect to the errors of the method and to round-off errors [Cor1].

We show in Subsect. 4.3.2 that symbolic images of a dynamical system generated by partitions of its phase space [Os1] can be applied to approximate its Morse spectrum [Os2].

In Sect. 4.4, we investigate qualitative properties of semi-implicit discretizations of (0.7). In Subsect. 4.4.1, we study finite-dimensional diffeomorphisms generated by discretizations and the global attractors for these diffeomorphisms [Ei1]. It is shown that, for a generic nonlinearity $f(u)$, these global attractors have Morse-Smale structure, hence we can apply the theory of shadowing for structurally stable systems (Chap. 2).

In Subsect. 4.4.2, we apply shadowing results obtained in Sect. 3.4 to give explicit estimates (in terms of time and space steps) for the differences between approximate and exact solutions on unbounded time intervals [Lar4].

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Contents

List of Main Symbols

$\mathbb{R}$ – the set of real numbers;
$\mathbb{R}^n$ – the Euclidean n-space;
$\mathbb{Z}$ – the set of integer numbers;
$\mathbb{Z}_+ = \{k \in \mathbb{Z} : k \geq 0\}$;
$\mathbb{N}$ – the set of natural numbers;
$\mathbf{C}$ – the set of complex numbers;
$\mathrm{GL}(n, \mathbb{R})$ ($\mathrm{GL}(n, \mathbf{C})$) – the group of invertible linear transformations of $\mathbb{R}^n$ (respectively, of $\mathbf{C}^n$);
I is the identity operator (or the unit matrix);
For a set A in a topological space, $\overline{A}$ is the closure of A, $\mathrm{Int}A$ is the interior of A, and ∂A is the boundary of A;
For sets A, B in a metric space (X, r), $N_a(A)$ is the a-heighborhood of A,

$$\mathrm{diam}A = \sup_{x,y \in A} r(x, y)$$

and

$$\mathrm{dist}(A, B) = \inf_{x \in A, y \in B} r(x, y);$$

For a linear mapping A,

$$||A|| = \sup_{|v|=1} |Av|$$

is the operator norm of A;
For a Banach space $\mathcal{B}$, $\mathcal{B}(r)$ is the closed ball of radius r centered at 0;
For a smooth manifold M, T_xM is the tangent space of M at x and TM is the tangent bundle of M;
For a smooth mapping f, $Df(x)$ is the derivative of f at x;
$:=$ means "equal by definition".