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White Noise Calculus and Fock Space

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Introduction

The white noise calculus (or analysis) was launched out by Hida [1] in 1975 with his lecture notes on generalized Brownian functionals. This new approach toward an infinite dimensional analysis was deeply motivated by Lévy [1] who considerably developed functional analysis on $L^2(0,1)$ and actually analysis of Brownian functionals. The root of white noise calculus is to switch a functional of Brownian motion $f(B(t); t \in \mathbb{R})$ with one of white noise $\phi(\dot{B}(t); t \in \mathbb{R})$, where $\dot{B}(t)$ is a time derivative of a Brownian motion $B(t)$. Although each Brownian path $B(t)$ is not smooth enough, $\dot{B}(t)$ is thought of as a generalized stochastic process and ϕ is realized as a generalized white noise functional in our language. We may thereby regard $\{\dot{B}(t)\}$ as a collection of infinitely many independent random variables and hence a coordinate system of an infinite dimensional space.

The mathematical framework of the white noise calculus is based upon an infinite dimensional analogue of the Schwartz distribution theory, where the role of the Lebesgue measure on $\mathbb{R}^n$ is played by the Gaussian measure μ on the dual of a certain nuclear space E. In the classical case where $\dot{B}(t)$ is formulated, we take $E = \mathcal{S}(\mathbb{R})$ and the Gaussian measure μ on E^* defined by the characteristic functional:

$$\exp\left(-\frac{|\xi|^2}{2}\right) = \int_{E^*} e^{i\langle x, \xi\rangle}\,\mu(dx), \qquad \xi \in E,$$

where $|\xi|$ is the usual L^2-norm of ξ. Then the Hilbert space $(L^2) = L^2(E^*, \mu)$ is canonically isomorphic to the (Boson) Fock space over $L^2(\mathbb{R})$ through the Wiener-Itô-Segal isomorphism and links the test and generalized functionals. Namely, in a specific way (called *standard construction*) we construct a nuclear Fréchet space (E) densely and continuously imbedded in (L^2), and by duality we obtain a Gelfand triple:

$$(E) \subset (L^2) = L^2(E^*, \mu) \subset (E)^*.$$

An element in (E) is a *test white noise functional* and hence an element in $(E)^*$ is a *generalized white noise functional.* The above picture is easily understood as a direct analogy of $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ which is a frame of the Schwartz distribution theory. Then, $\dot{B}(t) = x(t)$, $x \in E^*$, gives us a realization of the time derivative of a Brownian motion and, in fact, $x \mapsto x(t)$ becomes a generalized white noise functional for each fixed $t \in \mathbb{R}$.

In our actual discussion we do not restrict ourselves to the case of $E = \mathcal{S}(\mathbb{R})$ and $H = L^2(\mathbb{R})$ but deal with a more general function space on a topological space T. Typically T is a time-parameter space and is often taken to be a more general topological space where quantum field theory may be formulated. Again $\{x(t); t \in T\}$ is considered as a coordinate system of E^* intuitively. In fact, within our framework we may discuss not only functionals in $\{x(t); t \in T\}$ but also operators derived from this coordinate system. The coordinate differential operator $\partial_t = \partial/\partial x(t)$ is well defined as a continuous derivation on (E). We have also multiplication operators by coordinate functions $x(t)$, which are, in fact, operators from (E) into $(E)^*$. Furthermore, ∂_t^* is a continuous linear operator on $(E)^*$. The operators ∂_t and ∂_t^* correspond respectively to an *annihilation operator* and a *creation operator* at a point $t \in T$ and they satisfy the so-called canonical commutation relation in a generalized sense. The above mentioned formulation was consolidated in the basic works of Kubo and Takenaka [1]-[4] and has been widely accepted.

The main purpose of these lecture notes is to develop operator theory on white noise functionals as well as to offer a systematic introduction to white noise calculus. From that point of view it is most remarkable that we are free from smeared creation and annihilation operators. In other words, ∂_t and ∂_t^* are not operator-valued distributions but usual operators for themselves. This leads us to an integral kernel operator:

$$\Xi_{l,m}(\kappa) = \int_{T^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m)\partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m,$$

where κ is a *distribution* in $l+m$ variables. The use of distributions as integral kernels allows us to discuss a large class of operators on Fock space. In fact, *every* continuous operator Ξ from (E) into $(E)^*$ admits a unique decomposition into a sum of integral kernel operators:

$$\Xi\phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \qquad \phi \in (E),$$

where the series converges in $(E)^*$. Moreover, if Ξ is a continuous operator from (E) into itself, the series converges in (E). In the process we investigate precise norm estimates of such operators and obtain a method of reconstructing an operator from its symbol. The above expression is called *Fock expansion* and will play a key role in our discussion.

Although applications of white noise calculus are widely spreading, the present lecture notes are strongly oriented toward *infinite dimensional harmonic analysis*. The clue to go on is found in the following three topics: (i) infinite dimensional rotation group; (ii) Laplacians; (iii) Fourier transform. Being almost as new as the white noise calculus, they have been so far discussed somehow separately. Since the very beginning of the development Hida has emphasized the importance of the infinite dimensional rotation group $O(E; H)$, that is, the group of automorphisms of the Gelfand triple $E \subset H \subset E^*$. In fact, it played an interesting role in the study of symmetry of Brownian motion and Gaussian random fields. There are various candidates for infinite dimensional Laplacians which possess some typical properties of a finite dimensional Laplacian. So far the Gross Laplacian Δ_G, the number operator N and the Lévy Laplacian Δ_L have been found to be important in white noise calculus, though the Lévy Laplacian is not discussed in these lecture notes. As for Fourier transform, among some candidates that have been discussed Kuo's Fourier transform (simply called the *Fourier transform* hereafter) has been found well suited to white noise calculus.

In these lecture notes the above listed three subjects are treated systematically by means of our operator calculus and are found closely related to each other. For example, the Gross Laplacian Δ_G and the number operator N are characterized by their rotation-invariance. The Fourier transform intertwines the coordinate differential operators and coordinate multiplication oprators just as in the case of finite dimension and, this property actually characterizes the Fourier transform. Moreover, the Fourier transform is imbedded in a one-parameter transformation group of the generalized white noise functionals (called the *Fourier-Mehler transform*) and its infinitesimal generator is expressed with Δ_G and N. These results would suggest a fruitful application of white noise calculus to infinite dimensional harmonic analysis. It is also expected that our operator calculus is useful in some problems in quantum field theory and quantum probability.

As is well known, a lot of efforts to develop distribution theories on an infinite dimensional space equipped with Gaussian measure have been made by many authors. In fact, mathematical study of Brownian motion or equivalently of white noise is now one of the most important and vital fields of mathematics toward infinite dimensional analysis.

Since the main purpose is to develop an operator theory on white noise functionals, the present lecture notes are mostly based on a functional analytic point of view rather than probability theory or stochastic analysis. In Chapter 1 we survey some fundamentals in functional analysis required during the main discussion and propose a notion of a standard countably Hilbert space which makes the discussion clearer. The purpose of Chapter 2 is to establish the well-known Wiener-Itô-Segal isomorphism between $L^2(E^*, \mu)$ and the Fock space. Chapter 3 is devoted to a study of generalized white noise functionals. In Chapter 4 we develop an operator theory on white noise functionals, or equivalently on Fock space, in terms of Hida's differential operators ∂_t and their duals ∂_t^*. By means of the operator theory we discuss in Chapter 5 a few topics toward harmonic analysis including first order differential operators, the number operator, the Gross Laplacians, infinite dimensional rotation group, Fourier transform and certain one-parameter transformation groups. Chapter 6 is added after finishing the first draft of these lecture notes. We discuss integral-sum kernel operators, the finite dimensional calculus derived from our framework and a generalization to cover vector-valued white noise functionals. These topics are expected to open a new area in infinite dimensional analysis.

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My basic references have been among others the works of Hida-Potthoff [1], Kuo [7], [9], Lee [3] and Yan [4] which I appreciated highly. The readers are recommended to consult the recently published monograph Hida-Kuo-Potthoff-Streit [1] which contains different topics and various applications. It will complement our discussion certainly.

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