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Equivariant K-Theory and Freeness of Group Actions on C*-Algebras

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Preface

This book is about equivariant K-theory and K-theoretic conditions for freeness of actions of compact Lie groups on C^*-algebras. The introduction, explaining in detail the motivation for this work, is followed by two primarily expository chapters, one each on equivariant K-theory for C^*-algebras and equivariant KK-theory for C^*-algebras. The remaining six chapters contain the results of the author's research on K-theoretic conditions for freeness of actions on C^*-algebras. We assume throughout familiarity with the theory of C^*-algebras, including crossed product C^*-algebras and ordinary (that is, not equivariant) K-theory of C^*-algebras.

Our work is motivated by the observation that, given an action of a compact Lie group on a compact Hausdorff space, one can determine whether the action is free solely by examining the equivariant K-theory of the space for the given action. (Details, with proofs, are given in chapter 1.) Now the category of compact Hausdorff spaces is contravariantly equivalent to the category of commutative unital C^*-algebras via the functor assigning to each space the algebra of continuous complex-valued functions on the space. Therefore the relation between equivariant K-theory and freeness can be interpreted as an assertion about actions of compact Lie groups on commutative unital C^*-algebras. In view of the recent successes of noncommutative algebraic topology, in which general C^*-algebras are regarded as "noncommutative locally compact spaces," we are naturally led to try to generalize the results mentioned above to general C^*-algebras. However, no completely satisfactory notion of freeness of an action on a C^*-algebra is known. We therefore define and study conditions on the equivariant K-theory of a C^*-algebra which, in case the algebra is commutative, imply that the action on the underlying space is free.

Chapters 2 and 3 develop the necessary background material on C^*-algebraic equivariant K-theory and KK-theory respectively. Neither chapter depends significantly on the rest of the book. Chapter 2 consists mostly of material which has previously appeared only in somewhat condensed form, or which has never been published but straightforwardly generalizes ordinary K-theory. It is fairly comprehensive; however, there are some results purely about equivariant K-theory elsewhere, especially in sections 5.1 and 6.1-6.4. Previous knowledge of ordinary K-theory for C^*-algebras is assumed, but is used only at a few points, most notably for the six term exact sequence and Bott periodicity. Chapter 3 develops equivariant KK-theory from Cuntz' quasihomomorphism point of view. Most of the material has appeared previously but again only in very condensed form. We prove only the basic facts, up to the product and Bott periodicity, omitting however the construction of the six term exact sequences, for which we refer to a paper of Cuntz and Skandalis. Again, some additional material can be found elsewhere, particularly in sections 5.1 and 9.7.

In chapter 4, we define our K-theoretic notions of freeness. There and in the next two chapters we consider consistency theorems, the analogs of such facts as the freeness of the restriction of a free action to an invariant subspace or a closed subgroup. Most of the appropriate statements are either easily proved or, in bad cases, easily disproved. Two topics, namely actions of subgroups and actions on tensor products, present greater difficulties. Each of these topics gets a chapter to itself, and the gaps between our theorems and our counterexamples are larger than was generally the case in chapter 4.

Chapter 7 is devoted to the relation between our K-theoretic notions of freeness and previously known measures of freeness, especially Kishimoto's strong Connes spectrum. Our conditions are, unfortunately, trivially satisfied by a trivial action on a simple C^*-algebra whose K-groups are all zero. (This kind of difficulty cannot arise in the context of spaces.) However, if the K-theory of the C^*-algebra is sufficiently nontrivial, and if the group is sufficiently small, then K-theoretic freeness does imply other forms of freeness.

The remaining two chapters consider the implications of our K-theoretic conditions for freeness for actions on two special classes of C^*-algebras, namely type I and AF algebras. In both cases, we obtain analytic characterizations of several of our K-theoretic freeness conditions. In the type I case, one of our conditions is shown to be equivalent to freeness of the corresponding action on the primitive ideal space. For AF algebras, we obtain results of a somewhat different nature but which should not be surprising in view of known results about the ordinary K-theory of AF algebras. Our results here have led us to hope that there might be a good analytic notion of freeness which implies K-theoretic freeness in general and coincides with it on these special classes of C^*-algebras. However, we have made no attempt to investigate this question.

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