

# Lecture Notes in Mathematics

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Masahiro Shiota

Nash Manifolds

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## PREFACE

The purpose of these Lecture Notes is to construct a theory of real manifolds equipped with "algebraic" structures. Real nonsingular algebraic varieties display apparently singular phenomena, and this suggests we should not expect a unified theory to emerge from them. As we shall see, however, the objects we consider in these Notes form a broader class than the real nonsingular algebraic varieties.

A function on  $\mathbb{R}^n$  is algebraic if and only if it is analytic and if the graph is semialgebraic in  $\mathbb{R}^n \times \mathbb{R}$ . We call such a function a  $C^\omega$  Nash function. We also call a  $C^r$  map with semialgebraic graph,  $0 \leq r \leq \omega$ , a  $C^r$  Nash map. Then we define a  $C^r$  Nash manifold by sticking finite semialgebraic open sets in  $\mathbb{R}^n$  together along semialgebraic open subsets by  $C^r$  Nash diffeomorphisms. It is the category of  $C^r$  Nash manifolds and  $C^r$  Nash maps that we are interested in.

The compact case was already studied well. In [N] Nash showed that a compact  $C^1$  manifold  $M$  can be imbedded in a Euclidean space  $\mathbb{R}^n$  so that the image is a  $C^\omega$  Nash submanifold of  $\mathbb{R}^n$ . He proved also that such a  $C^\omega$  Nash manifold structure on  $M$  is unique up to  $C^\omega$  Nash diffeomorphism. Hence we can endow a compact  $C^1$  manifold with "algebraic" properties, which appears to contribute to differential topology. Indeed there are two applications [A-M] and [K]. Hoping for more applications, Palais developed a theory of compact affine  $C^\omega$  Nash manifolds in [Pa].

We extend the object from  $C^\omega$  Nash manifolds to  $C^r$  Nash manifolds. This is because there always exists a partition of unity of  $C^r$  Nash class if  $0 \leq r < \infty$ , a theorem of approximation of a  $C^r$  Nash map by a  $C^\omega$  Nash map holds true, and because the similarity and the contrast between the  $C^0$  Nash category and the  $C^\omega$  Nash category are clear and interesting. Our theory is based mainly on results in [Sh<sub>1</sub>], ..., [Sh<sub>5</sub>] and [S-Y].

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