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Peter Jipsen Henry Rose

Varieties of Lattices

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Authors

Peter Jipsen
Department of Mathematics
Iowa State University
Ames, Iowa 50011, USA

Henry Rose
Department of Mathematics
University of Cape Town
7700 Rondebosch
Cape Town, South Africa

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Synopsis

An interesting problem in universal algebra is the connection between the internal structure of an algebra and the identities which it satisfies. The study of varieties of algebras provides some insight into this problem. Here we are concerned mainly with lattice varieties, about which a wealth of information has been obtained in the last twenty years.

We begin with some preliminary results from universal algebra and lattice theory. The second chapter presents some properties of the lattice of all lattice subvarieties. Here we also discuss the important notion of a splitting pair of varieties and give several characterizations of the associated splitting lattice. The more detailed study of lattice varieties splits naturally into the study of modular lattice varieties and nonmodular lattice varieties, dealt with in the third and fourth chapter respectively. Among the results discussed there are Freese's theorem that the variety of all modular lattices is not generated by its finite members, and several results concerning the question which varieties cover a given variety. The fifth chapter contains a proof of Baker's finite basis theorem and some results about the join of finitely based lattice varieties. Included in the final chapter is a characterization of the amalgamation classes of certain congruence distributive varieties and the result that there are only three lattice varieties which have the amalgamation property.

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We dedicate this monograph to our supervisor Bjarni Jónsson.

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Introduction

The study of lattice varieties evolved out of the study of varieties in general, which was initiated by Garrett Birkhoff in the 1930's. He derived the first significant results in this subject, and further developments by Alfred Tarski and later, for congruence distributive varieties, by Bjarni Jónsson, laid the groundwork for many of the results about lattice varieties. During the same period, investigations in projective geometry and modular lattices, by Richard Dedekind, John von Neumann, Garrett Birkhoff, George Grätzer, Bjarni Jónsson and others, generated a wealth of information about these structures, which was used by Kirby Baker and Rudolf Wille to obtain some structural results about the lattice of all modular subvarieties. Nonmodular varieties were considered by Ralph McKenzie, and a paper of his published in 1972 stimulated a lot of research in this direction.

Since then the efforts of many people have advanced the subject of lattice varieties in several directions, and many interesting results have been obtained. The purpose of this book is to present a selection of these results in a (more or less) self-contained framework and uniform notation.

In Chapter 1 we recall some preliminary results from the general study of varieties of algebras, and some basic results about congruences on lattices. This chapter also serves to introduce most of the notation which we use subsequently.

Chapter 2 contains some general results about the structure of the lattice Λ of all lattice subvarieties and about the important concept of "splitting". We present several characterizations of splitting lattices and Alan Day's result that splitting lattices generate all lattices. These results are applied in Chapter 4 and 6.

Chapters 3 – 6 each begin with an introduction in which we mention the important results that fall under the heading of the chapter.

Chapter 3 then proceeds with a review of projective spaces and the coordinatization of (complemented) modular lattices. These concepts are used to prove the result of Ralph Freese, that the finite modular lattices do not generate all modular lattices. In the second part of the chapter we give some structural results about covering relations between modular varieties.

In Chapter 4 we concentrate on nonmodular varieties. A characterization of semidistributive varieties is followed by several technical lemmas which lead up to an essentially complete description of the "almost distributive" part of Λ. We derive the result of Bjarni Jónsson and Ivan Rival, that the smallest nonmodular variety has exactly 16 covers, and conclude the chapter with results of Henry Rose about covering chains of join-irreducible semidistributive varieties.

Chapter 5 is concerned with the question which varieties are finitely based. A proof of Kirby Baker's finite basis theorem is followed by an example of a nonfinitely based variety, and a discussion about when the join of two finitely based varieties is again finitely based.

In Chapter 6 we study amalgamation in lattice varieties, and the amalgamation property. The first half of the chapter contains a characterization of the amalgamation class of certain congruence distributive varieties, and in the remaining part we prove that there are only three lattice varieties that have the amalgamation property.

By no means can this monograph be regarded as a full account of the subject of lattice varieties. In particular, the concept of a congruence variety (i.e. the lattice variety generated by the congruence lattices of the members of some variety of algebras) is not included, partly to avoid making this monograph too extensive, and partly because it was felt that this notion is somewhat removed from the topic and requires a wider background of universal algebra.

For the basic concepts and facts from lattice theory and universal algebra we refer the reader to the books of George Grätzer [**GLT**], [**UA**] and Peter Crawley and Robert P. Dilworth [**ATL**]. However, we denote the join of two elements a and b in a lattice by $a+b$ (rather than $a \vee b$) and the meet by $a \cdot b$, or simply ab (instead of $a \wedge b$; for the meet of two congruences we use the symbol $\cap$). When using this plus, dot notation, it is traditionally assumed that the meet operation has priority over the join operation, which reduces the apparent complexity of a lattice expression.

As a final remark, when we consider results that are applicable to wider classes of algebras (not only to lattices) then we aim to state and prove them in an appropriate general form.