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Invariant Manifolds, Entropy and Billiards; Smooth Maps with Singularities

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1. INTRODUCTION

During the past twenty-five years the hyperbolic properties of smooth dynamical systems (i.e. of diffeomorphisms and flows) were studied in the ergodic theory of such systems in a more and more general framework (see $[\text{Ano}]_{1,2}$, [Sma], [Nit], [Bri], $[\text{Kat}]_1$, $[\text{Pes}]_{1,3}$, $[\text{Rue}]_{2,3}$). The detailed historical survey of the hyperbolicity and its role in the ergodic theory up to 1967 is given in $[\text{Ano}]_2$, Chapter 1.

One of the most important features of smooth dynamical systems showing behavior of hyperbolic type is the existence of invariant families of stable and unstable manifolds and their so called "absolute continuity". The most general theorem concerning the existence and the absolute continuity of such families has been proved by Ya. B. Pesin ($[\text{Pes}]_{1,2}$).

The final results of this theory give a partial description of the ergodic properties of a smooth dynamical system with respect to an absolutely continuous invariant measure in terms of the Lyapunov characteristic exponents. One of the most striking of the many important consequences of these results described in $[\text{Pes}]_{2,3}$ is the so called Pesin entropy formula which expresses the entropy of a smooth dynamical system through its Lyapunov characteristic exponents.

Our first main purpose is to generalize Pesin's results to a broad class of dynamical systems with singularities and at the same time to fill gaps and correct errors in Pesin's proof of absolute continuity of families of invariant manifolds ($[\text{Pes}]_1$, Sec. 3). We followed Pesin's scheme very closely and this may at least partly explain the length of our presentation and heaviness of details, especially in Part II. Parts I and II contain the theory of stable (and unstable) invariant manifolds in our more general situation and correspond to the context of $[\text{Pes}]_1$. At the end of Part II we also prove an infinite dimensional counterpart of Pesin's results from $[\text{Pes}]_1$.

The motivation for our generalization lies in the fact that some important dynamical systems occurring in classical mechanics (for example, the motion of the system of rigid balls with elastic collisions) do have singularities. Some of these systems (including the example mentioned) can be reduced to so-called billiard systems. Briefly speaking, a billiard system describes the motion of a point mass within a Riemannian manifold with boundary with reflection from the boundary. Our general conditions on the singularities formulated in Sec. 1 of Part I grew out of an attempt to understand the nature of singularities in the billiard problem.

Since a Poincaré map (first-return map on a section) for a smooth flow usually has singularities, considering transformations with singularities may also provide a unified treatment of discrete time and continuous time dynamical systems.

In Part III we prove the below estimate for the Pesin entropy formula. This part reproduces with minor changes the paper $[\text{Led}]_1$, whose idea goes back to $[\text{Sin}]_1$ and $[\text{Pes}]_{2,3}$. This proof uses in an essential way the results of Parts I and II. Recently R. Mañe ($[\text{Mañ}]_1$) gave in smooth case an alternative very ingenious proof of the estimation from below. His proof avoids completely the use of invariant manifolds and it is substantially simpler than the proof along the Sinai-Pesin line. It seems that Mañe's method can be applied to our case.

The above entropy estimate proved in Part IV is largely independent of the rest of the book.

In $[\text{Pes}]_3$ Pesin derives from his results on invariant stable and unstable manifolds the description of ergodic properties of a smooth dynamical system on the invariant set with non-zero Lyapunov exponents. All his arguments with the sole exception of his proof of Bernoulli property literally apply to our case. It seems that the proof of Bernoulli property requires a somewhat stronger estimate of the Jacobian of the Poincaré map than the one obtained in Part II.

Results from $[\text{Kat}]_2$ concerning the connection between entropy and the growth of periodic points also hold in our situation assuming that the measure satisfies the conditions from Sec. 1 of Part I.

In Part V we study in great detail the singularities of the Poincaré map for plane billiards and show that the conditions from Sec. 1 of Part I are satisfied with respect to the natural absolutely continuous invariant measure for a broad class of such systems. This class includes all compact regions bounded by a finite number of convex and concave arcs of class C^3 and straight line intervals, with the extra assumption: every convex arc has the tangency of finite order with all its tangents. By the results of Parts III and IV the Pesin entropy formula is satisfied for such billiards. We do not know whether the above entropy estimate through the Lyapunov exponents holds for an <u>arbitrary</u> invariant measure for such a billiard. Let us notice that recently M. Wojtkowski ($[\text{Woj}]_{1,2}$) found an easy proof that for so-called Sinai-Bunimovich billiards the Lyapunov exponents are non-zero.

Resuming, one can say that in the present book we completed the lower right corner of the following diagram, The theory of Anosov systems and of the related systems as Axiom A systems, etc. ⟶ The theory of billiards of Sinai and Bunimovich

↓ ↓

Pesin Theory of diffeomorphisms of compact manifolds ⟶ Pesin Theory of mappings with singularities

A concise résumé of the main results of the present book can be found in [Str].

Other presentations of Pesin's theorem concerning the existence of invariant manifolds were given later by D. Ruelle ($[\text{Rue}]_1$) and A. Fathi, M. Herman and J.-C. Yoccoz ([Fat]). D. Ruelle has developed several generalizations of that theorem (non-invertible smooth maps, a class of infinite-dimensional maps ($[\text{Rue}]_{2,3}$)). R. Mañe has found another infinite-dimensional version of Pesin's theorem ($[\text{Mañ}]_2$).

The authors would like to point out their unequal participation in the preparation of this book. Almost all the text was actually written by the second author. The first author suggested the general plan of the work and worked out the arguments which allow us to overcome the presence of singularities in the construction of invariant manifolds and in the above entropy estimate. Naturally, we discussed together numerous questions concerning practically all subjects treated in the text.

The first draft of the theory described in the present book was presented by the second author in December 1978 at the Seminar of Mathematical Physics at IHES (Bures-sur Yvette, France). The material of this book represents a part of the "Thèse d' Etat" of the second author, defended 30 April 1982 at University Paris VI (France).

Our notations are very similar to those used by Pesin, but they are not the same.

Concerning the enumeration of formulas, theorems, etc, the first number indicates the section in which the given formula, theorem, etc., is contained. The lower Roman numeral indicates the part of the book. In the interior of the same parts, the Roman numerals are not marked.

Despite all our efforts, some mistakes can remain. We will be grateful to the readers kind enough to point them out.

Acknowledgments. This book owes very much to Dr. F. Ledrappier (CNRS, University Paris VI, France) and to Dr. F. Przytycki (Mathematical Institute of Polish Academy of Sciences, Warsaw).

Besides being a co-author of Part III, F. Ledrappier made numerous useful remarks concerning other topics treated in the book. In particular he played a very important role in the elaboration of the infinite dimensional case.

The role of F. Przytycki can hardly be overestimated. We owe him the final formulation of conditions characterizing our class of maps with singularities. In the previous versions conditions on the growth of the first derivative as well as of the growth of the two first derivatives of the inverse mapping near the singularities were assumed. Using ideas of F. Przytycki we were able to dispose of these conditions in Parts I-III and consequently to extend the class of mappings under consideration. We thank sincerely both of them.

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