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Emilio Bujalance
José J. Etayo
José M. Gamboa
Grzegorz Gromadzki

Automorphism Groups of Compact Bordered Klein Surfaces

A Combinatorial Approach

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Authors

Emilio Bujalance
Dpto. de Matemáticas Fundamentales
Facultad de Ciencias
Universidad a Distancia (UNED)
28040 Madrid, Spain

José Javier Etayo
José Manuel Gamboa
Dpto. de Algebra
Facultad de Matemáticas
Universidad Complutense
28040 Madrid, Spain

Grzegorz Gromadzki
Institute of Mathematics, WSP
Chodkiewicza 30
85-064 Bydgoszcz, Poland

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To Raquel (and our daughters
Raquel, M.Teresa, Carla and Lidia)
To Almudena
To Alicia
To Terenia

INTRODUCTION

Classical results on automorphism groups of complex algebraic curves.

Given a complex algebraic curve by means of its polynomial equations, it is very difficult to get information about its birational automorphisms, unless the curve is either rational or elliptic. For curves C of genus $p \geq 2$, Schwarz proved in 1879 the finiteness of the group Aut(C) of automorphisms of C, **[111]**. Afterwards Hurwitz, applying his famous ramification formula, showed that $|\mathrm{Aut}(C)| \leq 84(p-1)$, **[69]**. By means of classical methods of complex algebraic geometry, Klein showed that $|\mathrm{Aut}(C)| \leq 48$ if $p=2$. Then, Gordan proved that $|\mathrm{Aut}(C)| \leq 120$ for $p=4$, **[51]**, and Wiman, who carefully studied the cases $2 \leq p \leq 6$, in particular established that $|\mathrm{Aut}(C)| \leq 192$ for $p=5$ and $|\mathrm{Aut}(C)| < 420$ for $p=6$, **[127]**, **[128]**. Hence in case $p=2,4,5,6$, Hurwitz's bound is not attained. In case $p=3$, there is only one curve of genus 3 with $168=84(3-1)$ automorphisms: Klein's quartic $x^3y+y^3+x=0$.

Apart from these and other facts on curves of low genus, further results of a more general nature were known by the end of the last century. For instance, Wiman proved that the order of each automorphism of C is always $\leq 2(2p+1)$; he also studied hyperelliptic curves in detail. Every curve of genus $p=2$ is hyperelliptic, and so it admits an involution. On contrary, for $p \geq 3$, curves with non-trivial automorphisms are exceptional; they constitute the singular locus of the moduli space of curves of genus p, and this explains the interest of this topic.

Complex algebraic curves and Riemann surfaces.

As observed by Riemann, the birational geometry of (irreducible) complex algebraic curves can be studied in a transcendental way. Indeed, algebraic function fields in one variable over $\mathbb{C}$ are nothing but fields of meromorphic functions on compact Riemann surfaces. Hence, groups of birational automorphisms of complex algebraic curves are the same as automorphism groups of compact Riemann surfaces.

This new point of view propelled the theory ahead, in the early sixties, with the remarkable work of Macbeath. After Poincare **[106]**, it was well known that each compact Riemann surface S of (algebraic) genus $p \geq 2$ can be represented as an orbit space H/Γ of the upper half complex plane H. Here H is endowed with the conformal structure induced by the group Ω of Möbius transformations, and the acting group Γ is a fuchsian group, *i.e.*, a discrete subgroup of Ω. The group Γ can be chosen with no elements of finite order. With this representation at hand, Macbeath proved that a finite group G is a group of automorphisms of S if and only if $G=\Gamma'/\Gamma$ for another fuchsian group Γ', **[82]**. This was a new, combinatorial, topological, group-theoretical method, to adress questions on groups of automorphisms.

The general strategy for a better understanding of automorphism groups of a compact Riemann surface S of genus $p \geq 2$ is explained by Macbeath **[82]**, and paraphrased by Accola, **[1]**. First one looks for some group G of automorphisms of S. Let T_0 be the set of branching points in $S_0=S/G$ of the n-sheeted covering $\pi:S \longrightarrow S_0$, and let F_0 be the fundamental group of $S_0 \backslash T_0$. Let K be the kernel of the natural homomorphism from F_0 into the n-th symmetric group. Then a given automorphism $f_0 \in Aut(S_0)$ lifts to some $f \in Aut(S)$ if and only if f_0 restricts to a permutation on T_0 and $f_0^*(K)=K$. As a consequence, if $Aut(S_0)$ contains a subgroup with m elements verifying these conditions, then Aut(S) contains a subgroup of order mn. Notice that this method heavily relies on a good choice of G in order to work well with $Aut(S_0)$. The techniques used here are those of Weierstrass points and of fuchsian groups. A large number of results can be proved with the aid of the first method, but many others require the second one.

The set W of Weierstrass points of S is finite and each $f \in Aut(S)$ restricts to a permutation of W. This permutation determines completely f when S is not hyperelliptic and completely, up to the canonical involution, when S is hyperelliptic. This gives another proof of the finiteness of Aut(S). The method of Weierstrass points is specially fruitful to provide geometrical information about the behaviour of an automorphism h of S. For example, there is an equality relating the order and number of fixed points of h with the genera of S and S/h, **[7]**, **[42]**, **[47]**, **[80]**.

The theory of fuchsian groups is the most powerful in order to investigate the structure of automorphism groups of complex algebraic curves. However, the situation is not fully satisfactory, in the following sense: given such a curve C we can represent its associated Riemann surface S_C in the form H/Γ_C. Unfortunately, except for its algebraic structure, we do not have a good knowledge of the fuchsian group Γ_C. In particular, although Γ_C reflects some geometrical properties of C as hyperellipticity or q-gonality, there is no known relation between Γ_C and any algebraic equations defining C. This explains why results on groups of automorphisms of complex algebraic curves obtained by means of fuchsian groups theory are not effective. Sometimes it is possible to prove the existence of a curve C with certain geometric properties whose group of automorphisms is of a given type, but we cannot find explicit algebraic equations for C.

Some results on automorphisms of Riemann surfaces.

Let us retrieve some significant facts in this area, obtained by means of fuchsian group theory. Hurwitz's ramification formula can be read as $|\Gamma'/\Gamma|=\mu(\Gamma)/\mu(\Gamma')$, where Γ is a surface fuchsian normal subgroup of the fuchsian group Γ' and μ represents the area of a fundamental region of the corresponding group. If $S=H/\Gamma$ has genus p, then $\mu(\Gamma)=4\pi(p-1)$ while, by Siegel's theorem **[113]**, $\mu(\Gamma')\geq\pi/21$. Hence $|Aut(S)|\leq 84(p-1)$ as announced and the equality is only attained if Γ' is the fuchsian triangular group M=(2,3,7). In such a way, the study of Hurwitz groups (=groups of order 84(p-1) acting on surfaces of genus $p\geq 2$), becomes the study of finite factors of M.

Using this, Macbeath proved the existence of Hurwitz groups for infinitely many values of p - for p=7 in **[85]** and for $p=2m^6+1$ in **[83]**. The non-existence of Hurwitz groups for infinitely many values of p was also proved in the latter paper. The same kind of arguments led Accola **[1]** and Maclachlan **[89]** to show that for every $p\geq 2$ there is a surface of genus p with 8(p+1) automorphisms; actually, there are infinitely many of such surfaces. Besides, Greenberg showed that every finite group is the group of automorphisms of some surface **[53]**, and Harvey computed the minimum genus of the surfaces admitting an automorphism of a given order, **[61]**. As a consequence, he found a new proof of Wiman's bound. Among the vast literature on the subject we must quote here the papers by Cohen, **[37]**, **[38]**, Greenberg **[54]**, Sah **[110]** and Zomorrodian **[131]**.

Real algebraic curves and Klein surfaces.

Up to now we only have been concerned with *complex* algebraic geometry. However, what happens if the ground field is the field $\mathbb{R}$ of real numbers?. This was overlooked for a long time but in the last two decades many beautiful mathematics have been developed in the real setting, demanding specific tools for this new field of research (*cf.* the books of Brumfiel **[10]**, Delfs-Knebusch **[41]**, Bochnak-Coste-Roy **[8]** and Knebusch **[74]**).

A particular question in this area is the study of groups of birational automorphisms of real algebraic curves. As in the complex case, very few is known about Aut(C) from the algebraic equations defining C except when C has genus 0 - Lüroth's theorem - or genus 1, see Alling **[4]**. Even the recent methods in effective algebra and computational geometry do not give yet a satisfactory answer to this question.

In order to work in an analogous way to the complex case, one is forced to enlarge the category of Riemann surfaces. This raises two different problems. Firstly, since nonorientable surfaces do not admit any analytic structure, a more general notion is needed. The suitable one is dianalyticity, which includes analytic and antianalytic maps; the former preserve the orientation, the latter reverse it and both are conformal. Then, orientable and nonorientable surfaces, with or without boundary, admit a dianalytic structure. These are Klein surfaces, introduced by Alling and Greenleaf **[5]**, following up ideas by Klein. Of course, Riemann surfaces are precisely the orientable, unbordered Klein surfaces.

Secondly, it was necessary to represent algebraic function fields in one variable over $\mathbb{R}$ as fields of "meromorphic functions" on some objects. These objects are exactly Klein surfaces. Moreover, *real* fields (=fields in which -1 is not a sum of squares) are the meromorphic function fields of *bordered* surfaces. Thus the categories of compact bordered Klein surfaces and of real (irreducible) algebraic curves are equivalent. This functorial equivalence is necessary to apply the methods described before to problems in real geometry. We explain it in pure real algebra terms in the appendix at the end of the book.

Hence, for the study of birational automorphisms of real algebraic curves we can focus on compact Klein surfaces S of algebraic genus $p \geq 2$. The notion of real Weierstrass points has not been exploited yet, and so combinatorics is the only accessible approach. In his unpublished thesis **[108]**, Preston proved the real counterpart of Poincaré's uniformization theorem, **[106]**: S is the quotient of H under the action of a non-euclidean crystallographic (NEC in short) group, that is, a discrete subgroup of the extended modular group. This NEC group can be assumed having no orientation preserving mapping of finite order. This theorem, together with the classification of NEC groups due to Macbeath **[86]** and Wilkie **[126]**, opened the door to the combinatorial approach to groups of automorphisms presented here.

Since 1975 many papers have appeared studying groups of automorphisms of compact Klein surfaces of genus ≥ 2, analogous to the ones described before on Riemann surfaces. We shall either present or report them along the book.

It is remarkable that no technique is developed to study groups of automorphisms of algebraic curves defined over real closed ground fields distinct to $\mathbb{R}$. Only in some cases, Tarski's Transfer Principle allows us to translate the results, but not the proofs, from the "true reals" to arbitrary real closed fields. As an example, one can see [23].

Contents of this book.

The general problem we analyze can be stated as follows: given a class $\mathscr{G}$ of finite groups and a class $\mathscr{K}$ of compact Klein surfaces of algebraic genus $p \geq 2$, under what conditions do a surface S in $\mathscr{K}$ and a group G in $\mathscr{G}$ exist, such that G acts as a (or the full) group of automorphisms on S? This is essentially the concern of chapters 3 to 6, which constitute the core of the book.

In chapter 3, $\mathscr{G}$ is the class of cyclic groups and we decide, in a computable way, whether a given natural number is the order of an automorphism of some surface of fixed topological type. Moreover, if the surface is orientable, we can decide whether such an automorphism preserves or reverses the given orientation. As a consequence, we determine the minimum genus of surfaces admitting an automorphism of a given order and the maximum order of an automorphism of a surface of a given genus, with all precisions concerning orientability.

In chapter 4 we consider several classes of finite groups: soluble, supersoluble, nilpotent, p-groups, abelian. For each of these classes $\mathscr{G}$ we compute an upper bound $N(p, \mathscr{G})$ for the order of a group of automorphisms G of a surface of genus p, provided $G \in \mathscr{G}$. Moreover, we calculate the topological type of the surfaces attaining this maximal group of automorphisms and the algebraic structure of this group. Most results are original except a few which were proved first by May using different techniques. We present here a unified treatment of the problem. We must remark that, as technical lemmata for the proofs, we obtain several results on abstract group theory, as presentations of supersoluble and nilpotent groups.

In chapter 5 we look at the family $\mathscr{K}$ of surfaces with nonempty connected boundary. Groups acting as groups of automorphisms in them are cyclic or dihedral and we determine for a fixed group G of this type the existence of a surface $S \in \mathscr{K}$ with $Aut(S) = G$, in terms of the genus and orientability of S.

The same problem is solved in chapter 6 for the family of hyperelliptic surfaces. Here groups of automorphisms are extensions by $\mathbb{Z}_2$ of cyclic and dihedral groups.

As said before, if a surface S is written as H/Γ for a surface NEC group Γ, then $Aut(S) = \Lambda/\Gamma$ where Λ is the normalizer of Γ in the extended modular group. Hence, the computation of Aut(S) involves the knowledge of this normalizer, which is a rather difficult problem. We avoid it by using Teichmüller spaces theory to prove that maximal signatures correspond to maximal NEC groups. Then we call to the lists of non-maximal signatures of Singerman **[115]** and Bujalance **[13]**. The results of chapters 5 and 6 follow from this, together with specific procedures highly depending on the features of the involved surfaces - *e.g.* the existence of a unique central involution in the hyperelliptic case.

The remaining chapters have a different nature. In chapter 0 we describe briefly Klein surfaces, NEC groups and some general properties on Teichmüller spaces that are needed later to compute full groups of automorphisms. For the proofs of some results in this chapter, the reader is refered to the original papers where these fundamental facts were established.

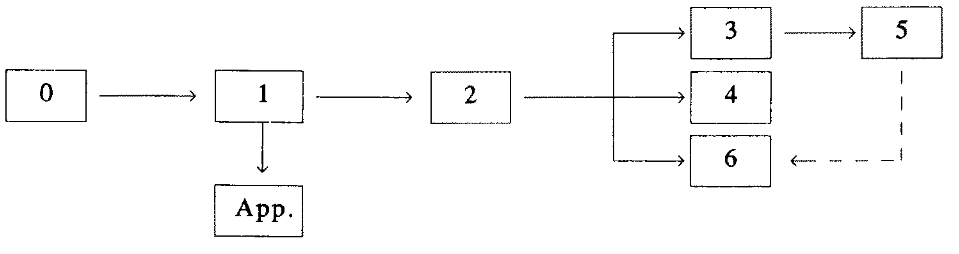
Chapter 1 is a detailed presentation of the fundamental results of Preston quoted above, and a theorem by May saying that groups of automorphisms of H/Γ are quotients Λ/Γ where Λ is another NEC group. This naturally leads to study, in chapter 2, the relation between the presentations of an NEC group Λ and of its normal subgroup Γ, using combinatorial methods based mainly in surgery on a fundamental region of Λ. Most of this is classical, but was scattered in the literature. So we develop it here in a unified and precise fashion for later use in the combinatorial study of automorphism groups.

All in all, the reader familiar with the prerequisites mentioned in chapter 0, will find the book self-contained.

Of course, part of the material belongs to other people, and the proper attributions appear in the historical notes included at the end of each chapter.

We have also included both a subject and a symbol indices at the end of the text.

Interdependence of chapters



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The authors
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