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with contributions by J. E. McClure

Equivariant Stable Homotopy Theory

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Preface

Our primary purpose in this volume is to establish the foundations of equivariant stable homotopy theory. To this end, we shall construct a stable homotopy category of G-spectra enjoying all of the good properties one might reasonably expect, where G is a compact Lie group. We shall use this category to study equivariant duality, equivariant transfer, the Burnside ring, and related topics in equivariant homology and cohomology theory.

This volume originated as a sequel to the volume "H_∞ ring spectra and their applications" in this series [20]. However, our goals changed as work progressed, and most of this volume is now wholly independent of [20]. In fact, we have two essentially disjoint motives for undertaking this study. On the one hand, we are interested in equivariant homotopy theory, the algebraic topology of spaces with group actions, as a fascinating subject of study in its own right. On the other hand, we are interested in equivariant homotopy theory as a tool for obtaining useful information in classical nonequivariant homotopy theory. This division of motivation is reflected in a division of material into two halves. The first half, chapters I-V, is primarily addressed to the reader interested in equivariant theory. The second half, chapters VI-X, is primarily addressed to the reader interested in nonequivariant applications. It gives the construction and analysis of extended powers of spectra that served as the starting point for [20]. It also gives a systematic study of generalized Thom spectra. With a very few minor and peripheral exceptions, the second half depends only on chapter I and the first four sections of chapter II from the first half. The reader is referred to [105] for a very brief guided tour of some of the high spots of the second half.

Chapter I gives the more elementary features of the equivariant stable category, such as the theory of G-CW spectra and a desuspension theorem allowing for desuspension of G-spectra by all representations of G in the given ambient "indexing universe". Chapter II gives the construction of smash products and function G-spectra. It also gives various change of universe and change of groups theorems. Chapter III gives a reasonably comprehensive treatment of equivariant duality theory, including Spanier-Whitehead, Atiyah, and Poincaré duality. Chapter IV studies transfer maps associated to equivariant bundles, with emphasis on their calculational behavior in cohomology. Chapter V studies the Burnside ring and its role in equivariant stable homotopy theory. It includes various related splitting theorems in equivariant homology and cohomology theory.

Although we have encountered quite a few new phenomena, our main goals in the first half have been the equivariant generalization of known nonequivariant results and the generalization and sharpening of known equivariant results. We therefore owe ideas and material to numerous other mathematicians. Our general debt to the work of Boardman [13,14] and Adams [1] in nonequivariant stable homotopy theory will be apparent throughout. The idea for a key proof in chapter I is due to Hauschild. The main change of groups theorems in chapter II are generalizations of results of Wirthmuller [144] and Adams [3], and the study of subquotient cohomology theories in II§9 is based on ideas of Costenoble.

Our debts are particularly large in chapters III, IV, and V. Our treatment of duality is largely based on ideas in the lovely paper [47] of Dold and Puppe and on (nonequivariant) details in the papers [63,64,65] of their students Henn and Hommel; equivariant duality was first studied by Wirthmuller [145]. Our treatment of transfer naturally owes much to the basic work of Becker and Gottlieb [10,11] and Dold [46], and transfer was first studied equivariantly by Nishida [117] and Waner [141]. Our IV§6 is a reexposition and equivariant generalization of Feshbach's work [53,54] on the double coset formula, and he cleared away our confusion on several points. While our initial definitions are a bit different, a good deal of chapter V is a reexposition in our context of tom Dieck's pioneering work [38-44] on the Burnside ring of a compact Lie group and the splitting of equivariant stable homotopy. This chapter also includes new proofs and generalizations of results of Araki [4].

A word about our level of generality is in order. We don't restrict to finite groups since, for the most part, relatively little simplification would result. We don't generalize beyond compact Lie groups because we believe that only the most formal and elementary portions of equivariant stable homotopy theory would then be available. The point is that, in all of our work, the depth and interest lies in the interplay between homotopy theory and representation theory. Technically, part of the point is that the cohomology theories represented by our G-spectra are RO(G)-graded and not just Z-graded. This implies huge amounts of algebraic structure which would be invisible in more formal and less specific homotopical contexts.

While a great deal of our work concerns equivariant cohomology theory, we have not given a systematic study here. Lewis, McClure, and I have used the equivariant stable category to invent "ordinary RO(G)-graded cohomology theories" [88], and the three of us and Waner are preparing a more thorough account [90]. (Hauschild, Waner, and I are also preparing an account of equivariant infinite loop space theory, which is less directly impinged upon by this volume.)

Chapters VI-VIII establish rigorous foundations for the earlier volume [20], which we shall refer to as $[H_\infty]$ here. That volume presupposed extended powers $D_jE = E\Sigma_j \ltimes_{\Sigma_j} E^{(j)}$ of spectra with various good properties. There E was a nonequivariant spectrum, but our construction will apply equally well to G-spectra E for any compact Lie group G.

In fact, extended powers result by specialization of what is probably the most fundamental construction in equivariant stable homotopy theory, namely the twisted half-smash product $X \ltimes E$ of a G-space X and a G-spectrum E. (The "twisting" is encoded by changes of universe continuously parametrized by X.) This construction is presented in chapter VI, although various special cases will have been encountered earlier.

We develop a theory of "operad ring G-spectra" and in particular construct free operad ring G-spectra in chapter VII. When G is finite, special cases give approximations of iterated loop G-spaces $\Omega^V\Sigma^V X$, and we obtain equivariant generalizations of Snaith's stable splittings of spaces $\Omega^n\Sigma^n X$.

We prove some homological properties of nonequivariant extended powers that were used in $[H_\infty]$ in chapter VIII.

Chapters IX and X give a careful treatment of the Thom spectra associated to maps into stable classifying spaces. These have been used extensively in recent years, and many people have felt a need for a detailed foundational study. In chapter IX, we work nonequivariantly and concentrate on technical problems arising in the context of spherical fibrations (as opposed to vector bundles). In chapter X, we work equivariantly but restrict ourselves to the context of G-vector bundles. There result two specializations to the context of nonequivariant vector bundles, the second of which is the more useful since it deals naturally with elements of $KO(X)$ of arbitrary virtual dimension.

We must again acknowledge our debts to other mathematicians. We owe various details to Bruner, Elmendorf, and McClure. The paper of Tsuchiya [138] gave an early first approximation of our definitions of extended powers and H_∞ ring spectra. As explained at the end of VII§2, Robinson's A_∞ ring spectra [124] fit naturally into our context. The proof of the splitting theorem in VII§5 is that taught us by Ralph Cohen [34]. We owe the formulations of some of our results on Thom spectra to Boardman [12] and of others to Mahowald [93], whose work led to our detailed study of these objects.

Each chapter of this book has an introduction summarizing its main ideas and results. There is a preamble comparing our approach to the nonequivariant stable category with earlier ones, and there is an appendix giving some of the more esoteric proofs. References are generally by name (Lemma 5.4) when to results in the same chapter and by number (II.5.4) when to results in other chapters.

Finally, I should say a word about the genesis and authorship of this volume. Chapter VIII and part of chapter VI are based on Steinberger's thesis [133], and chapter VII started from unpublished 1978 notes of his. Chapter IX and the Appendix are based on Lewis' thesis [83], and the definition and axiomatization of the transfer in chapter IV are simplifications of his work in [85]. Chapter V incorporates material from unpublished 1980 notes of McClure. All of the rest of the equivariant material is later joint work of Lewis and myself.

The authorship of the several chapters is as follows.

Chapters I through IV: Lewis and May

Chapter V: Lewis, May, and McClure

Chapters VI and VII: Lewis, May, and Steinberger

Chapter VIII: May and Steinberger

Chapter IX: Lewis

Chapter X: Lewis and May

The Appendix and the indices were prepared by Lewis.

J. Peter May
June 20, 1985

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