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The Numerical Solution of Differential-Algebraic Systems by Runge-Kutta Methods

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Preface

The term differential-algebraic equation has been coined to comprise differential equations with constraints (differential equations on manifolds) and singular implicit differential equations. Such problems arise and have to be solved in a variety of applications, e.g., constrained mechanical systems, fluid dynamics, chemical reaction kinetics, simulation of electrical networks, and control engineering. From a more theoretical viewpoint, the study of differential-algebraic problems gives insight into the behaviour of numerical methods for stiff ordinary differential equations. As a consequence, this subject has attracted the interest of many engineers and mathematicians in the last years.

The purpose of these lecture notes is to give a self-contained and comprehensive exposition of the numerical solution of differential-algebraic systems arising in applications, when treated by Runge-Kutta methods, here included also extrapolation methods. While multistep methods (BDF) have been considered since the early seventies (Gear (1971)), the study of Runge-Kutta methods for differential-algebraic systems has begun only a few years ago. Runge-Kutta methods also have interesting computational and theoretical properties. They combine high order with good stability, allow a simple step size selection, are self-starting and have advantages in parallel computing.

The first two sections are introductory and review differential-algebraic problems and Runge-Kutta methods for their numerical solution. In Sections 3 to 6 we study existence and uniqueness of the numerical solution, influence of perturbations, local error and convergence, and asymptotic expansions. We investigate in Sections 7 and 8 the convergence of simplified Newton iterations for the arising nonlinear systems, and the problems of local error estimation and inconsistent starting values. In the final sections we describe a FORTRAN program and apply it to several concrete examples. The sections end with notes which relate the results to the existing literature. Most of the presented material has not been published previously.

We have tried to treat the subject in its various aspects ranging from theory via numerical analysis to implementation and applications. Many of the presented ideas and techniques are not restricted to Runge-Kutta methods, but can also be applied to other integration methods, such as linearly implicit methods and multistep methods.

These lecture notes have their origin in a one-semester graduate course given by one of the authors at the University of Rennes, and in a series of seminars at the University of Geneva, in the years 1987 and 1988.

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