

# Lecture Notes in Computer Science

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Edited by G. Goos and J. Hartmanis

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I. Herman

# The Use of Projective Geometry in Computer Graphics

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Series Editors

Gerhard Goos  
Universität Karlsruhe  
Postfach 69 80  
Vincenz-Priessnitz-Straße 1  
W-7500 Karlsruhe, FRG

Juris Hartmanis  
Department of Computer Science  
Cornell University  
5148 Upson Hall  
Ithaca, NY 14853, USA

Author

Ivan Herman  
Centre of Mathematics and Computer Science  
Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

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## Preface

This book has its history. In 1986, I got involved with some of my friends in the implementation of a GKS-3D package at a small firm called Insotec Consult, in Munich, Germany. We did not have too much time, so we tried to produce a straightforward implementation of the Standard as fast as we could and we planned to optimize it in a later stage. We already had the common experience of a GKS-2D implementation, so the task seemed to be relatively easy. To implement the 3D output pipeline, we just had to program some appropriate matrix-matrix and matrix-vector multiplications and we soon started the first toy test-program, to navigate among randomly positioned wire-framed objects in space. The output seemed to be right at a first glance: we saw what we expected. However, in some cases some unexpected lines appeared on the screen in a fairly chaotic manner which we just could not explain to ourselves. Being self-critical enough, we set down to find the bug in our code by running through it several times, but without success. It was only after several days' of work that we begun to suspect that implementing a 3D pipeline is not that simple after all.

Of course, we did not do our homework properly; the notion of 'external lines' or, as it is called in this book, the 'W-Wraparound' (which turned out to be the reason for our problems) had already been presented in the literature before. However, not having easy access to literature in our small software house we had to find a solution for ourselves. This work had unexpected results for all of us; it made us realize that the computer graphics community makes astonishingly little use of the notions of projective geometry in solving its problems in spite of the fact that the very essence of 3D computer graphics is closely related to this classical branch of mathematics. Looking at the problem with fresh eyes has proven to be advantageous after all; it has led us to some really new and interesting results which have proven to be of a general interest, going beyond the particular problem which had triggered it. This story was the start for me of a long-term activity in trying to adapt some projective geometry results for the purposes of computer graphics; it led to a series of publications and to my PhD thesis which I defended in 1990 at the University of Leiden, in the Netherlands. This thesis has essentially provided the material for this book.

Such a small book obviously cannot and does not cover all the problems of a 3D output pipeline as it appears in practice, nor does it describe all possible applications of projective geometry in computer graphics. Instead, it concentrates on some of its special and very much algorithmic aspects which are related first of all to the different transformations used in computer graphics. However, all specialists or students of computer graphics who want to understand the underlying mathematic principles of 3D graphics systems or want to participate in the implementation of a new one can get, I hope, some inspiration for such work.

A hidden, but also very important aim of the book is to make clear what the

usual curricula in computer sciences seem not to emphasize enough: that higher-level mathematics — projective geometry is just an example — have a major role to play in computer graphics and computer science in general. A number of problems become easier to solve or just simply to describe provided that the appropriate mathematical tools are used. If this book succeeds in turning the attention of some of its readers toward mathematics again, it has achieved a major goal set for myself when planning to publish this work through Springer Verlag.

The mistake of the ‘missing homework’ was committed together with my colleague and friend, J. Reviczky, with whom I had a long and very fruitful cooperation before I decided to join the Center for Mathematics and Computer Sciences in Amsterdam; it is a pleasure for me to acknowledge his major role in the birth of this book. The ongoing discussions and common works with all my former colleagues of Insotec Consult, primarily J. Hübl, are also acknowledged and very much appreciated.

The text of this book was fleshed out, revised and shaped in discussions with my PhD advisers, namely Prof. F. Peters, Prof. R. Hubbold, Prof. G. Joubert, Prof. D. Duce and Prof. J. van den Bos. It is again a pleasure for me to acknowledge their help and encouraging remarks in the preparation of the manuscript. The outstanding facilities of the Center for Mathematics and Computer Sciences made it possible to produce the output reliable and the way I wanted. Finally, I am grateful to my wife, Eva, who really pushed me to do this work and helped me through difficult times; this work would never have been finished without her.

Amsterdam, October 1991, Ivan Herman

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