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The Selberg-Arthur Trace Formula

Based on Lectures by James Arthur

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Introduction

Since the appearence of A. Selberg's famous paper in 1956, it gradually became clear that the role of Selberg's trace formula in automorphic forms, number theory, and representation theory is fundamental.

Selberg's paper is essentially concerned with the derivation of a trace formula for the regular representation of the group $PSL_2(\mathbb{R})$, and the application of this formula in calculating the trace of the Hecke operators for that group.

The importance of Selberg's paper naturally raised the question of how to generalize this trace formula to groups of higher ranks, and how to apply the formula in many problems directly or indirectly related to automorphic forms. Here we shall explain very briefly the attempts which have been made over many years to advance Selberg's theory of the trace formula.

Firstly, regarding the generalizations, a basic idea had already been indicated in Selberg's paper itself. This idea was that the Eisenstein series has an important role to play in advancing the theory from PSL_2 to the higher rank groups. One reason for this was the need to characterize the continuous spectrum of the regular representation of a Lie group G into the Hilbert space $L^2(\Gamma\backslash G)$, for a discrete subgroup Γ of G. Indeed, as a result of Langlands' work on the Eisenstein series, the continuous spectrum of the spectral decomposition was fully characterized and a major step toward the generalization of Selberg's trace formula was thus taken. On the other hand, the Duflo-Labesse paper in the late sixties, and J. Arthur's work in the early seventies were important, since the former provided thorough rigorous proof through representation theory of the work of Selberg, and the latter took the first steps toward the generalization of the trace formula for the higher rank groups. Finally, in his papers written since 1974 which begins by [1], Arthur has achieved the generalization of Selberg's trace formula for the arbitrary reductive group G, defined over a number field F, for the case where Γ is an arithmetic congruence subgroup of G. Arthur's works on the generalization of Selberg's trace formula are firmly based on three topics: the geometry of conjugacy classes in $G(F)$; the spectral theory of the Eisenstein series; and the theory of weighted orbital integrals. Arthur's trace formula thus establishes an identity between distributions obtained by conjugacy classes (termed the geometric expansion), and distributions obtained by the Eisenstein series (termed the spectral expansion). Moreover, this identity of distributions is valid simultaneously at all places of F, i.e., it is given globally for the adèle group $G(\mathbf{A})$ and its discrete subgroup of F-rational points $G(F)$.

Secondly, with reference to the applications, Selberg himself has shown in his paper and in other places that one may use his trace formula to obtain results of great importance in number theory and automorphic forms. Probably the most fascinating applications of Selberg's trace formula are now to be found in the works of Langlands, among which one should mention the Base Change theory of $GL(2)$, and also in algebraic geometry, which includes the theory of Shimura varieties, elliptic curves and corresponding zeta functions. One can always expect to obtain more general results using Arthur's generalized trace formula; for example, the Base Change theory of Langlands has recently been generalized to $GL(n)$ by Arthur and Clozel using the generalized trace formula.

This book is based on Arthur's lectures at University of Toronto during the fall term of 1987, which I had the opportunity to attend. Similar lectures were also given by Arthur during Jan. - Feb. of 1986 at the Tata Institute of Fundamental Research. My reasons for writing up the notes are two-fold. In the first place, I found the lectures very stimulating for grasping the basic ideas of the trace formula, and secondly, the trace formula is a fundamental tool in the study of many questions in automorphic forms and representation theory. Therefore, this book may serve as a basic reference for the fundaments of the important theory developed by Arthur.

I was first introduced to Selberg's trace formula by Prof. I. Satake ([85], [100]). As a curious student in Selberg-Arthur theory, I wanted to learn more. The result is this book.

This book consists of eight chapters. Each chapter has a brief introduction. There are a few exercises which are presented informally. They are basically intended to summarize the proofs and to help the reader to observe the more fundamental material which lies behind the theory of trace formulas, or to show the application of the trace formulas.

I would like to render thanks to Professor R. P. Langlands for introducing me to the works of Arthur, while I was a member of the Institute for Advanced Study during the 1986-87 academic year. I would also like to thank James Arthur for giving me the chance to visit University of Toronto and learn from him some of the most advanced mathematics of our time. His readiness and willingness to answer my questions and his friendship were vitally important.

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