Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Non-Oscillation Domains of Differential Equations with Two Parameters

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

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Mathematics Subject Classification (1980): Primary: 34A30, 34C10, 45D05
Secondary: 34C15, 39A10, 39A12

ISBN 3-540-50078-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-50078-2 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

... Et si illa oblita fuerit, ego tamen non obliviscar tui,

Ecce in manibus mei descripsi te.

(Is. 49, 15-16)

Per Felice, Angelino ,Dulcineo e Michelino

In memoriam

Preface

The aim of these notes is to study the large-scale structure of the non-oscillation and disconjugacy domains of second order linear differential equations with two parameters and various extensions of the latter.

We were heavily influenced in this endeavor by a paper of Markus and Moore [Mo.2]. The applications to Hill's equation, Mathieu's equation, along with their discrete analogs, motivated many of the questions, some resolved, and some unresolved, in this work.

As we wished to consider linear systems of second order differential equations, Sturmian methods had to be avoided. Thus we chose to base the theory essentially on variational methods - For this reason many of the results herein will have analogs in higher dimensions (e.g., for Schrödinger operators) although we have not delved into this matter here.

The first author (ABM) wishes to thank the Department of Mathematics, University of Trondheim - N.T.H., for making possible his visit there during September 1984 and for the kind hospitality rendered while there. He also wishes to acknowledge with thanks the support of the Natural Sciences and Engineering Research Council of Canada in the form of a research grant and a University Research Fellowship. Finally, he acknowledges with thanks the assistance of France Prud'Homme and Manon Gauvreau in the typing of various versions of these notes and his colleagues David Handelman and Edward L. Cohen for their generous time in technical assistance on the Macintosh SE.

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Ottawa, March 1985 and May 1988.

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1. Introduction

The study of the solutions of equations of the form

$$y'' + (-\alpha + \beta B(x))y = 0, \tag{1}$$

where α, β are real parameters and x varies over some real interval is of widespread importance in various branches of pure and applied mathematics. For example, the Mathieu equation, which arises naturally in connection with the problem of vibrations of an elliptic membrane, is of the form (1) with $B(x) = \cos 2x$. On the other hand, Lamé's equation, which occurs in the theory of the potential of an ellipsoid, is of the same type but with B(x) being a Jacobi elliptic function. Various eigenvalue problems may be cast in the form (1) - Fixing α and allowing β to be the parameter we obtain a weighted Sturm-Liouville equation with a possibly sign indefinite B(x). Such equations have received much attention lately and we refer the interested reader to the survey [Mi.3] for further information on this subject. Of interest is also the case when β is a function of α thus allowing for a nonlinear dependence on the parameter in question.

We are interested in studying the qualitative and spectral properties of equations of the form (1). In particular, we consider the set of all pairs (α, β) for which (1) has a solution which is positive in the interior of the interval under consideration and study topological properties of this set [defined below]. Central to our investigations is the paper by R.A. Moore [Mo1], who discusses the connection between the non-oscillation and periodicity of solutions of the Hill-type equation (1), in the case when B is continuous, periodic of period one and has mean value equal to zero.

Equation (1) is said to be **disconjugate on** $(-\infty, \infty)$ if and only if every one of its nontrivial solutions has at most one zero in $(-\infty, \infty)$. It is said to be **non-oscillatory on** $(-\infty, \infty)$ if and only if everyone of its nontrivial solutions has at most a finite number of zeros in $(-\infty, \infty)$. The collection of all $(\alpha, \beta) \in \mathbf{R}^2$ for which (1) is disconjugate (resp. non-oscillatory) on $(-\infty, \infty)$ will be dubbed the **disconjugacy domain** (resp. **non-oscillation domain**) of (1), and denoted by D (resp. N). Moore [Mo1] showed that, in fact, $D = N$ and that N is a closed, convex and unbounded subset of the $\alpha\beta$-plane which we call **parameter space** and label it $\mathbf{R}^2$ for simplicity. The results in [Mo1] were complemented by a paper of L. Markus and R.A. Moore [Mo2] in which B(x), appearing in (1), is now a (Bohr) almost-periodic function [Be.1] (or, for any sequence $\tau_n \in \mathbf{R}$, the sequence $B(x + \tau_n)$ has a subsequence which converges uniformly on $(-\infty, \infty)$).

The closedness of D (and/or N) and its convexity are what we will term the large-scale properties of D (or N) for reasons which will become clear below.

In this monograph we are concerned with the above-mentioned large-scale properties of D (and N) for the more general equation

$$y'' + (-\alpha A(x) + \beta B(x))y = 0 \qquad (2)$$

on the closed half-line $I \equiv [0, \infty)$ (although many of the results herein are also valid on $(-\infty, \infty)$). In (2) we assume **minimal requirements** on A, B, in general, in the sense that, A, B:$[0, \infty) \to \mathbf{R}$, and A, B $\in L_1^{loc}(0, \infty)$, i.e., they are Lebesgue integrable on every compact subset of $[0, \infty)$. Thus we waive all types of "periodicity" assumptions on A, B, and examine the consequences on the large-scale properties of D and N.

It turns out that, in fact, D is always a closed and convex set which can even be a **bounded** set in $\mathbf{R}^2$. In general $D \subsetneq N$ and N is also convex though **not always** a closed set. (See §§ 2.1-2.2). In §2.3 we present general conditions on A, B which ensure that $D \subseteq N \subseteq H^+$ where $H+ = \{(\alpha, \beta): \alpha > 0\} \cup \{0,0\}$ as is the case for Hill's equation [Mo1].

The lack of periodicity-type assumptions on A, B usually has the effect of splitting D and N, however we will see that $D = N$ may occur even in the "non-periodic" case (see § 2.1).

We then apply the foregoing results to the Sturm-Liouville equation (§ 2.5) and its extensions (§ 2.6) to potentials in which the eigen-parameter occurs quadratically.

In §2.7 we pose the general question - When is $D = N$? We show that, in particular, if A, B are Stepanov almost-periodic functions [Be1], then $D = N$ (§2.8). Further extensions of this result to the class of Weyl almost-periodic or more generally, Besicovitch almost-periodic functions appears doubtful, (see [Be1] for terminology).

In §3 we review the notions of disconjugacy for second order vector differential equations of the form (2) where, generally, A, B are n × n real matrix-functions whose entries are $L_1^{loc}(I)$. We introduce the new concepts of strong-and weak-disconjugacy and study the large-scale properties of D in these cases. As is to be expected, when A, B are real symmetric, and "disconjugate" has its usual meaning, many of the results of §2 allow extensions to the vector case.

In §4 we analyze the large-scale properties of D and N corresponding to Volterra-Stieltjes integral equations [At1], [Mi2] as they include in their structure, the theory of differential equations and, furthermore, the theory of difference equations (in this case, three-term recurrence relations).

The techniques which allow these extensions are basically variational in nature, unlike the ones in [Mo1, Mo2] which relied upon variable change and the nature of B(x) in (1). In general one cannot rely upon Sturmian arguments. Thus it is possible, although we shall not delve into this matter here, to extend many of the results herein to the setting of elliptic partial differential equations with two parameters.