Prerequisites in Algebraic Topology the Nordfjordeid Summer School on Motivic Homotopy Theory

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Preface

The intention of this note is **NOT** to write an introductory textbook in algebraic topology!! Many excellent sources exist, let me only point to Hatcher's book [7] which is available online.

On the contrary, these notes (and the resulting lectures at the summer school on motivic homotopy theory) attempt to give a quick overview of the parts of algebraic topology, and in particular homotopy theory, which are needed in order to appreciate that side of motivic homotopy theory.

This means that we have to study spaces and spectra, but in a way which allows for new applications and interpretations. Unfortunately, this point of view is not predominant in most textbooks, and so even students with a first course algebraic topology might be hard put when exposed to this material without some background. In particular, we will use simplicial techniques. Good books on basic simplicial stuff include [6] and [17]. Good books on general model category theory include [21] (the original), [4], [9], and [8].

The first chapter gives a quick presentation of the classical situations where homotopy theory is much used. In the second chapter we make a more thorough study of the key example: simplicial sets. The reason I have chosen to use so much time on this particular example is twofold. Firstly, some of the results were chosen since they were going to be used later in Voevodsky's lectures. Secondly, some of the results were chosen since they are typical models for the kind of arguments that are used over and over again in this theory.

Then a short and inadequate presentation of model category theory appears (this actually was even less complete in the lectures since I was pressed for time at this point). Since spectra are so important to the theory and the set-up uses many of the general ideas of model categories, they close the chapter.

The fourth and last chapter gives one approach to motivic homotopy theory. We give a quick presentation of the category of motivic spaces and their spectra from a functorial point of view. Those not caring overly much for coherent smash-products can stay to the simpler theory, also explained. I stress that this is but one of many possible approaches, and is definitely colored by my own preferences. No proofs are provided, and the reader is referred to [3] for this particular approach, or to [12] for another using symmetric spectra (also discussed in [22]).

Prerequisites. The reader is assumed to be familiar with the basic aspects of point-set topology. For instance Chaps. 2 and 3 in [18] will be (more than) enough. Categorical language is used freely, and some readers will find comfort in having a copy of [16] within easy reach.

Caution. The sketch proofs spread around in these notes are only just that. Although they may seem to be worded like complete proofs, there may be claims put forth which in reality can be hard to establish. The intention has not been to give full proofs, but rather to expose the reader to the idea and methods useful for proving results of this type.

0.1 Notational Quirks

- $\mathbf{N}$: the monoid of natural numbers (contains zero).
- $\mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R} \subseteq \mathbf{C}$: the rings of integers, rationals, reals and complex numbers.
- $\mathcal{E}ns$: the category of sets.
- $\mathcal{A}b$: the category of abelian groups
- If $\mathcal{C}$ is a category and c and d are objects in $\mathcal{C}$, then $\mathcal{C}(c, d)$ is the set of morphisms in $\mathcal{C}$ from c to d.
- If $\mathcal{C}$ is a category then the *opposite category* $\mathcal{C}^{\mathrm{op}}$ is the category with the same objects, but all arrows reversed.
- If $f\colon a \to c$ and $f\colon b \to c$ are two maps in a category with coproducts, then the natural map $a \coprod b \to c$ is called $f + g$.