Lecture Notes in Mathematics 1742

Editors:
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Quantization and Non-holomorphic Modular Forms

Springer

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Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Unterberger, André:
Quantization and non-holomorphic modular forms / André Unterberger.
Berlin ; Heidelberg ; New York ; Barcelona ; Hong Kong ; London ;
Milan ; Paris ; Singapore ; Tokyo : Springer, 2000
(Lecture notes in mathematics ; 1742)
ISBN 3-540-67861-1

Mathematics Subject Classification (2000): 11F03, 11L05, 35S99, 44A12, 81S99

ISSN 0075-8434
ISBN 3-540-67861-1 Springer-Verlag Berlin Heidelberg New York

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Springer-Verlag Berlin Heidelberg New York
a member of BertelsmannSpringer Science+Business Media GmbH

© Springer-Verlag Berlin Heidelberg 2000
Printed in Germany

Typesetting: Camera-ready TEX output by the author
SPIN: 10724305 41/3142-543210 - Printed on acid-free paper

Foreword: instructions for use

The aim of this manuscript is to bring together quantization theory and the theory of non–holomorphic modular forms. It depends on a certain number of ideas from quantization theory, pseudodifferential analysis, partial differential equations and elementary harmonic analysis on one hand, from modular form theory on the other.

As it adresses itself to two rather distinct possible audiences, we include the present foreword, as an answer to the question "who might be interested in reading what ?". Still, let us stress that, from our point of view, trade between mathematical disciplines should be conducted on a reciprocal basis: we thus hope that some number theorists may view our present investigations not only, or even mostly, as a — well–founded or not — claim that pseudodifferential analysis has something to contribute to modular form theory but, also, as an invitation to join a possibly unfamiliar playground. That the game is far from being over will be shown at the end of this foreword.

The reader interested in modular form theory, but not in motivations from quantization theory, might be advised to jump from the introduction to section 7, in which the Rankin–Selberg unfolding method is extended to some fair degree: thanks to the hyperfunction concept, the extended method permits to recover the Roelcke–Selberg coefficients of any $f \in L^2(\Gamma\backslash\Pi)$ with $\Delta f \in L^2(\Gamma\backslash\Pi)$, without any decrease at infinity being required; Π is the upper half–plane, Δ is the Laplace–Beltrami operator on Π and $\Gamma = SL(2,\mathbb{Z})$. The method is then applied (sections 9, 11 and 12) to the case when f is the product, or the Poisson bracket, of two Eisenstein series $E_{\frac{1-\nu_1}{2}}$ and $E_{\frac{1-\nu_2}{2}}$. This brings (section 10) certain Dirichlet series in two variables $\zeta_n(s,t)$ and $\zeta_n^-(s,t)$, related to Kloosterman–Selberg series, into the picture. In section 14, the preceding results are generalized so far as the admissible values of ν_1 and ν_2 are concerned: also, the coefficients of the discrete terms in the decomposition of $f_1 f_2$ or $\{f_1, f_2\}$ are made explicit in terms of L–functions and Hecke's theory. In retrospect, this permits to analyze the complex continuation of the series $\zeta_n^{\pm}(s,t)$. A central section of this work is section 15, which describes some rather simple Dirichlet series in one variable (theorems 15.2 and 15.3), the poles of which yield the eigenvalues of Δ. It also lets (theorems 15.6 and 15.7) the Maass cusp–forms appear as the residues with respect to some complex parameter μ of some simple series, generalizing Eisenstein's. Now it is not a novel thing to describe the eigenvalues of Δ and the Maass cusp–forms (or their Fourier coefficients) as poles and residues of appropriate series: but our series $F_{\mu,\nu}$, closely modelled after Eisenstein's, look as simple as one might wish; also, our method, based on the idea that, *in some algebraic sense*, Eisenstein series alone should generate all non–holomorphic modular forms, may be new too. To our knowledge, previous results of a related nature regarding Maass cusp–forms were based either on the study of the Green's kernel of the automorphic Laplacian [**28, 20**], or on Selberg's non–holomorphic Poincaré series [**33**] (which occur here in section 20, for a quite distinct purpose). For the benefit of readers who would insist on delimiting exactly what is new in our results, we show (in the remark which follows the proof of proposition 14.6) that when $|\text{Re}\,(s-t)| > \frac{3}{2}$, the complex continuation of $\zeta_n(s,t)$ could be obtained as a consequence of known results concerning the Kloosterman–Selberg series: but this does not hold when (s,t) lies in the domain $|\text{Re}\,(s-t)| < 1$ crucial for our investigations in section 15.

What precedes covers most of what is contained in the present monograph concerning non–holomorphic modular forms viewed independently from other subjects, though sections 19–20 could also be viewed in this context. Actually, these are in part the result of our desire to understand, from an analyst's point of view, why quadratic numbers enter the Maass construction of cusp–forms for congruence subgroups of Γ, but they also owe part of their inspiration to classical potential theory, namely to the notion of single layer potential. Though it is difficult to get one's hands on cusp–forms in any concrete way, it is easy to construct automorphic functions which are regular and modular of the non–holomorphic type almost everywhere. For instance, Eisenstein series arise as such a kind of object, with singularities at infinity (and all Γ–equivalent points): substituting a finite point z^o for ∞, one is led to Hejhal–type pseudo–cusp–forms. This is especially interesting, number–theoretically, when z^o is taken as an imaginary quadratic point. We here show how to construct automorphic, almost modular, functions on Π, with mild singularities (discontinuities in the normal derivative) spread out on the (locally finite) collection of Γ–transforms of a line with extremities conjugate in some real quadratic extension of $\mathbb{Q}$.

One last construction which, we hope, can interest number theorists, is our generalization of the Rankin–Cohen products [**5**] of modular forms to the non–holomorphic case: this was a major starting point of the present work, and it is explained in the introduction which follows.

All the rest of the monograph is concerned with quantization theory: this may start as a combination of representation theory (a group G, a Hilbert space $\mathcal{H}$, a unitary representation π of G into $\mathcal{H}$) and pseudodifferential analysis (the prescription of a covariant *quantizing map* from some *phase space* X on which G acts to the space of operators on $\mathcal{H}$, or its inverse *symbol map* [**42**]). Arithmetic comes into play as soon as some arithmetic subgroup Γ of G has been fixed, and one tries to analyze the space, or hopefully algebra, of operators whose symbols are Γ–invariant "functions" on X.

This very rich structure is interesting for several reasons, including the following: two quite distinct phase spaces X and $\tilde{X}$ may carry symbolic calculi related to the same representation π [**40**]. In this way, identifying the two species of symbols of the same operator commuting with the group of unitaries $\pi(\Gamma)$ leads to an identification of automorphic objects living on X or $\tilde{X}$. For instance, $\mathbb{R}^2\backslash\{0\}$ (with ξ and $-\xi$ identified) and the upper half–plane Π are so related since (section 17) one can relate (parts of) the Weyl calculus on $L^2(\mathbb{R})$ to some of the symbolic calculi available with Π as a phase space. In this way, non–holomorphic modular forms on Π are identified with homogeneous Γ–invariant distributions on $\mathbb{R}^2\backslash\{0\}$ (on which Γ acts in a linear way).

This can also be done with the help of the Radon transformation (section 4) or elementary representation theory (sections 2, 3). However, as shown in section 13, only the point of view of pseudodifferential analysis permits a true understanding of the "correct" Hilbert space of automorphic (*i.e.*, Γ–invariant) distributions on $\mathbb{R}^2$: this is not a trivial problem because there is no fundamental domain for the action of Γ on $\mathbb{R}^2\backslash\{0\}$. Still, a number of explicit automorphic distributions are nice objects to play with: this is described in section 16, where attention is also paid, in this classical distribution setting, to constructions with a more adelic taste.

On the other hand, something which we had not expected beforehand occurs, and is described in section 18: namely, it is possible to identify automorphic distributions with Cauchy data for the Lax–Phillips scattering theory for the automorphic wave equation. This identification has two versions: a first one in which both species appear as boundary values for two related boundary–value problems in the three–dimensional light–cone, and a deeper one in which the connection is analyzed through quantization theory. It is part of our projects (actually well under way) to develop the automorphic Weyl calculus, in which automorphic distributions act as symbols: this calculus is very exotic in comparison to all known pseudodifferential analyses and, in preparation for it, section 5 contains a composition formula for Weyl symbols of the usual type which, we hope, readers well versed in pseudodifferential analysis may find interesting.

Considering the quite impressive point to which the scope of modular form theory has been extended in recent years, it is hoped, but we have not even started working on this, that a comparable analysis may be developed on $\mathbb{R}^n$, in connection with general arithmetic subgroups of the symplectic group.

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