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Integrable Systems in the realm of Algebraic Geometry

Second Edition



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Preface to the second edition

The present edition of this book, five years after the first edition, has been spiced with several recent results which fit naturally in the point of view that had been adapted in the original text and with some new examples and constructions that will help the reader to appreciate better our approach to integrable systems.

On this occasion I wish to thank my collaborators from the last five years, to wit Christina Birkenhake, Peter Bneken, t~ui Fernandes, Masoto Kimura, Vadim Kuznetsov, Marco Pedroni, Michael Penkava, Luis Piovan and Claude Roger for a fruitful interaction and for their warm friendship. Most of the results that have been added are taken from, or are inspired by, joint work with some of them; I acknowledge their permission to add these, sometimes unpublished, results.

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