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Rational Matrix Equations in Stochastic Control

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Introduction

> One often gets the impression that [the algebraic Riccati] equation in fact constitutes the bottleneck of all of linear system theory.
>
> J. C. Willems in [201]

Robust and optimal stabilization

A primary goal in linear control theory is stabilization, while plant variability and parameter uncertainties are major difficulties.
Clearly, a linear model can describe reality at best locally, i.e. only as long as the state of the modelled system is close to the equilibrium state of linearization. It is the task of stabilization to keep the state within a neighbourhood of the equilibrium. Nevertheless, any mathematical model can only describe reality approximately, since one always has to rely on simplifying assumptions and can never measure the parameters with absolute accuracy. Therefore, stabilization always has to take account of possibly time-varying parameter uncertainties in the linearized model.
A stabilization strategy is usually called robust, if it copes with parameter uncertainties of a certain class. The problem of robust stabilization is an active topic of current research and has produced a vast amount of work over the past about forty years with several thousands of publications and numerous textbooks.

An important branch of stabilization theory is concerned with optimal stabilization. Among all stabilizing controllers, one selects the one that minimizes a given cost functional. By an adequate choice of this cost functional, one tries to represent certain performance specifications of the controlled system. For instance, in linear quadratic control theory, the cost functional is a positive semidefinite quadratic form of the state and the control vector; it punishes

both large deviations of the state from the equilibrium and large values of the control input, which might be too energy-consuming or even destroy the system. A major motivation for the use of such cost functionals, however, is their mathematical tractability. They offer some means to parametrize and compare different controllers.
More recently, with the emergence of H^∞-control theory around 1980, there has been growing interest in indefinite quadratic cost functionals, which can be used to guarantee certain disturbance attenuation and robustness properties of the controlled system.

Parallel to the introduction of different cost functionals, there have always been attempts to model parameter uncertainties, and to find robust optimal stabilizers with respect to these uncertainties. Of course, there are many different possibilities to model uncertainty, and again the mathematical tractability has to be taken into account.
One common approach is to consider intervals of systems instead of one single nominal system. In other words, one specifies certain intervals that contain the system parameters and tries to solve the stabilization problem for a whole family of systems simultaneously. An outstanding result in this setup is the famous theorem by Kharitonov on interval polynomials [127], which has triggered off a whole line of research in this direction.
In contrast to the interval approach, which treats each system in the interval with equal probability, one also might assume certain statistical properties of the parameters to be known, such that some parameter values in fact are more likely to occur than others. This leads to stochastic models like stochastic differential equations. Actually, one might argue, whether a stochastic system should be regarded as a deterministic system with parameter uncertainty or as a different nominal model itself. Linear control systems with multiplicative white noise were introduced in the works of Wonham, e.g. [211, 213].
Another very successful approach models parameter uncertainty as a disturbance system in a feedback connection with the nominal system. Important classes of parameter uncertainties can be modelled in this way (e.g. [87]). Using this setup, one can apply disturbance attenuation techniques to design robust stabilizers. This constitutes the connection between H^∞-control and robust stabilization.
It is not surprising that the different concepts of parameter uncertainty can be combined. For instance, one can consider intervals of stochastic systems, or stochastic systems with a disturbance system in a feedback connection (e.g. [163]).

In the present work, we are concerned with optimal stabilization and disturbance attenuation problems for stochastic linear differential systems. These investigations were initiated by results of Hinrichsen and Pritchard in [107] and stand in a line with the above-mentioned robust stabilization problems.
Our main object is the theoretical discussion and numerical solution of generalized Lyapunov and Riccati equations arising in this context. The main tools

are the theory of resolvent positive operators and Newton's method. Let us briefly introduce these concepts.

Mean-square stability and resolvent positive operators

When dealing with stochastic systems, one has to make some decisions on the adequate interpretation of the model and an appropriate concept of stability. In fact, these issues are neither obvious nor completely settled in the literature. The problem is that, in order to keep the model tractable, one usually models the parameter uncertainty as white noise, which is an idealized stochastic process. The correct interpretation of this idealized process, however, depends on the true nature of the uncertainty.

In our investigation, we will consider stochastic differential equations of Itô type and use the concept of mean-square stability, because this interpretation fits well in a worst-case scenario – as we will see in Chapter 1. Moreover, it leads to stability criteria that are very similar to those known from the deterministic case. While in the deterministic case stability can be judged from a standard Lyapunov equation

$$A^*X + XA = Y \ ,$$

we have to consider a generalized Lyapunov equation

$$A^*X + XA + \sum_{i=1}^{N} A_0^{i*} X A_0^i = Y$$

in the stochastic case. We will interpret this generalized Lyapunov equation as a standard Lyapunov equation perturbed by some positive operator. Furthermore, we will see that the sum of a Lyapunov operator and a positive operator belongs to a special class of operators, called resolvent positive operators. Properties of resolvent positive operators are of central importance in stability and stabilization problems for stochastic linear systems. Therefore, we will spend quite some effort in the analysis of resolvent positive operators and Lyapunov operators. In particular, we will have to deal with the spectral theory of positive linear operators on a vector space ordered by a convex cone.

Rational matrix equations and Newton's method

Stabilization problems for deterministic linear systems in continuous time lead to the famous algebraic Riccati equation

$$0 = A^*X + XA + P_0 - (S_0 + XB)Q_0^{-1}(S_0^* + B^*X) \ ,$$

where the Hermitian matrix

$$M = \begin{bmatrix} P_0 & S_0 \\ S_0^* & Q_0 \end{bmatrix}$$

determines the cost functional.
The counterpart in stochastic control is a rational matrix equation of the form

$$0 = A^*X + XA + P_0 + \sum_{i=1}^{N} A_0^{i*}XA_0^i \; - \; \left(S_0 + XB + \sum_{i=1}^{N} A_0^{i*}XB_0^i\right) \times \left(Q_0 + \sum_{i=1}^{N} B_0^{i*}XB_0^i\right)^{-1} \left(S_0^* + B^*X + \sum_{i=1}^{N} B_0^{i*}XA_0^i\right) .$$

By analogy, we will address this equation as a (generalized) Riccati equation. The difference between the standard and the generalized Riccati equation mirrors the difference between the standard and the generalized Lyapunov equation on a higher level. While the derivative of a standard Riccati operator is a standard Lyapunov operator, the derivative of a generalized Riccati operator is a generalized Lyapunov operator. This observation – together with the properties of resolvent positive operators – turns out to be fruitful both for a theoretical analysis of solutions to the generalized Riccati equation and for their iterative computation by Newton's method.

Outline of the book

In Chapter 1, we introduce stochastic models and discuss some special properties which are relevant with respect to robustness issues. Our main goal in this chapter, however, is to motivate our notions of stability, stabilizability and detectability and to clarify their relation to the generalized Lyapunov and Riccati equation, which are in the center of our interest. Moreover, we produce a number of examples which serve as illustrations in the following chapters.

In Chapter 2, we are concerned with optimal stabilization and disturbance attenuation problems for stochastic linear systems. We reformulate these problems in terms of generalized Riccati inequalities. Some results from the literature are extended and presented in a slightly more general situation.

Chapter 3 is devoted to the study of resolvent positive operators. While, on the one hand, these considerations prepare the stage for the following chapters, we think, on the other hand, that the results in this chapter are of independent value. Therefore, we discuss resolvent positive operators in some more detail than is needed in our study of generalized Riccati equations. Major contributions of this chapter are results on the representation of certain operators between matrix spaces and on the numerical solution of linear equations with resolvent positive operators.

Chapter 4 contains some of our main results. These are non-local convergence results for Newton's method and modified Newton iterations applied to concave operators with resolvent positive derivatives. Moreover, we generalize a result on the use of double Newton steps. We have chosen the most general setup for our method of proof to work and formulated the result for the case of an ordered Banach space. For illustration we have included some applications which are not related to stochastic control theory.

Finally, in Chapter 5 we apply our results to solve different types of generalized Riccati equations. To this end, we introduce the notion of a dual generalized Riccati operator. The main technical problem in Chapter 5 consists in showing that the dual operator possesses certain concavity properties. Altogether, we obtain rather complete existence results for stabilizing solutions of generalized algebraic Riccati equations arising in optimal stabilization problems for regular stochastic linear systems. Concluding this chapter we resume the discussion of some of the examples presented in the first chapter and illustrate our theoretical results by numerical examples.

In the appendix, we have collected some facts on Hermitian matrices and Schur-complements, which we use frequently.

Mathematical background

Basically, we can distinguish between five mathematical foundation pillars which our work is built on and which correspond to the partition into chapters. Of course, these mathematical fields cannot be separated strictly, and from a more detached point of view, one might just see them as topics in a larger field. But to give an orientation let us name these pillars and trace there recent origins.

Firstly, we bear on the stability theory for stochastic differential equations. While the modern notion of stochastic integrals and differential equations can be traced back to the works of Kolmogorov 1931, Andronov, Vitt and Pontryagin 1934, Itô 1946, Gikhman 1955 and Stratonovich 1964 [136, 3, 117, 118, 81, 188] roughly within the years 1930–1960, the corresponding notions of stability seem to appear first in the seminal paper [126] by Kats and Krasovskij 1961 and have been worked out mainly by Khasminskij in the 1960s, whose results are collected in [130]. Other important sources are the work of Kushner 1967, collected in [143] and the collection of papers [35] edited by Curtain 1972. More recent references on the topic of stochastic differential equations are the books by Gikhman and Skorokhod 1972, Arnold 1973, Friedman 1975, Krylov 1980 and 1995, Ikeda and Watanabe 1981, Gard 1988, Karatzas and Shreve 1991, Da Prato and Zabczyk 1992, Kloeden and Platen 1995, and Oeksendal 1998, [82, 6, 73, 141, 142, 115, 80, 125, 38, 134, 156].

Secondly, we use the fundamental concepts of linear systems theory, such as linear state-space systems, feedback connection, optimal control, stabilizability, controllability, detectability and observability. These notions mainly date back to the profound work of Kalman, e.g. [119, 120, 121]. Further important sources are the textbooks by Brockett 1970, Rosenbrock 1970, Anderson and Moore 1971, Kwakernaak and Sivan 1972, Wonham 1979, Knobloch and Kwakernaak 1985, and Sontag 1998, [22, 170, 2, 144, 214, 135, 184], to name but a few. The more recent development of robust control theory has found its way e.g. into the books of Francis 1987, Ackermann 1993, Başar and Bernhard

1995, Green and Limebeer 1995, Zhou, Doyle and Glover 1995, Hassibi, Sayed and Kailath 1999, Chen 2000, Dullerud and Paganini 2000, Trentelman, Stoorvogel and Hautus 2001, [70, 9, 87, 220, 96, 25, 60, 192]. We have already mentioned that the particular topic of stochastic linear control systems originates in the work of Wonham to be found e.g. in [213]. Mean-square stabilization problems for systems with state and input dependent noise have been discussed e.g. in papers by Sagirow 1970, Haussmann 1971, McLane 1971, Kleinman 1976, Willems and Willems 1976, Bismut 1976, Phillis 1983, Bernstein 1987, Bernstein and Hyland 1988, Sasagawa 1989, Tessitore 1992, Drăgan, Morozan and Halanay 1992, [171, 97, 150, 133, 204, 17, 164, 16, 15, 175, 190, 59]. Recent contributions to the field have been the results of El Bouhtouri and Pritchard 1993, Morozan 1995, Drăgan, Halanay and Stoica 1996, Hinrichsen and Pritchard 1996 and 1998, Biswas 1998, El Bouhtouri, Hinrichsen and Pritchard 1999, and Petersen, Ugrinovskii and Savkin 2000 [63, 154, 57, 106, 107, 18, 62, 163] on robust control of stochastic linear systems, as well as the results of Yong and Zhou 1999 [218] on control problems with indefinite input weight cost.

Thirdly, we rely on the spectral theory of positive linear operators in ordered vector spaces, which originates with the works of Perron 1907 and Frobenius 1908, [162, 74]. A major contribution to this field was the comprehensive paper [140] by Krein and Rutman 1950, who extended the results of Perron and Frobenius to a general setting of positive operators on ordered vector spaces. These results were developed further e.g. in a number of papers by Krasnoselskij and Schaefer. Good references are the monographes by Krasnoselskij 1964, Schaefer 1971, Berman, Neumann and Stern, 1989, Krasnoselskij, Lifshits, and Sobolev 1989, Berman and Plemmons 1994, and the survey paper by Vandergraft 1968, [138, 177, 13, 139, 14, 197]. Results on resolvent positive operators, which are of particular importance for our purpose, have been obtained by Schneider 1965, Elsner 1970 and 1974, Arendt 1987, and Fischer, Hinrichsen and Son 1998 in [180, 65, 66, 5, 68]. Some of these results will be generalized slightly. Moreover, we will have to deal with special classes of operators between matrix spaces, which have attracted quite some interest during the last 30 years. The relevant references will be given later.

Fourthly, we build upon earlier results on Newton's method applied to operator equations in ordered Banach spaces. This theory originates with the work of Kantorovich 1948 in [122], which can e.g. be found in the textbook [123] by Kantorovich and Akilov 1964. Kantorovich's results were developed further in various directions. Of major importance for us are the paper [196] of Vandergraft 1967, who made use of inverse positive operators and the notion of convexity to prove the convergence of a Newton sequence, and the paper [131] of Kleinman 1968, who was the first to prove a non-local convergence result for Newton's method applied to a Riccati equation. Kleinman's result was extended in a series of papers by Wonham 1968, Hewer 1971, Coppel 1974, Guo and Lancaster 1998, [212, 102, 34, 94] and others. As a main result, we will give a very general form of this non-local convergence theorem, which applies at once to Riccati equations from deterministic and stochastic control both in continuous and discrete time.

Finally, we make contributions to the vast field of algebraic Riccati equations, where we have to deal both with the existing results and the standard tools used in this area. Like the modern theory of linear control systems, the theory of matrix Riccati equations originates with the work of Kalman 1960 [119]. Since then it has grown into a very active, independent field of research. It is impossible to give complete account of all directions pursued in this field; we refer to the collection of papers [19] edited by Bittanti, Laub and Willems 1991, and the monographes, [167, 151, 145, 116] by Reid 1972, Mehrmann 1991, Lancaster and Rodman 1995, and Ionescu, Oara and Weiss 1999. The role of algebraic Riccati equations and inequalities in H^∞-optimal control has been the topic of e.g. the paper [221] by Zhou and Khargonekar 1998, and Scherer's thesis 1990, [178]. Furthermore, we name a few papers that are closely related to our investigations. Firstly, we have the outstanding paper by Willems 1971, and subsequently the works of Coppel 1974, Molinari 1977, Churilov 1978, Shayman 1983, and Gohberg, Lancaster and Rodman 1986, [201, 34, 152, 31, 182, 183, 84] on the existence of largest and stabilizing solutions. Monotonicity properties of largest solutions are of particular interest for us and have been discussed in papers by Wimmer 1976–1992, Ran and Vreugdenhil 1988, Freiling and Ionescu 2001 and Clements and Wimmer 2001, [207, 208, 210, 166, 71, 32]. Generalized Riccati equations of the type discussed here have been considered e.g. in the work of Wonham 1968, de Souza and Fragoso 1990, Tessitore 1992 and 1994, Freiling, Jank and Abou-Kandil 1996, Drăgan, Halanay and Stoica 1997, Fragoso, Costa and de Souza 1998, Hochhaus, and partly Reurings [212, 52, 190, 191, 72, 58, 69, 111, 168].

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