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1189

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Fine Topology Methods in Real Analysis and Potential Theory

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P r e f a c e

There are two well-established seminars at the Department of Mathematical Analysis of the Charles University Prague. One is the seminar on Potential Theory and the other the seminar on Modern Theory of Real Functions. It has been established that some of the problems subjected to study in these seminars are closely related resulting in the combined investigation of the very foundations and interrelations of the

The first version of the manuscript was prepared in April 1982 in order to serve as a complement to lectures at Erlangen and Eichstätt Universities. Since that time the work has been greatly extended and is now quite voluminous. It is only fair to admit that the manuscript is not free of misprints and shortcomings and does not contain all sources dealing with the subject. Any useful commentary and supplementary contributions would be gratefully received by the authors.

As many persons cooperated in the preparatory work on the manuscript let us mention at least some of those who extended their generous help in support of the authors. Let us quote amongst others our colleagues O.John, D.Preiss, J.Veselý and students M.Chlebík, J.Hrdina, V.Kelar, T.Schütz, V.Šverák and R.Thomas.

At this occasion let us express our gratitude and indebtness to Mrs. Erika Ritter from Erlangen who has devoted the best care to the final re-typing of the text. Insertion of mathematical symbols was done by the authors who themselves are responsible for all the possible mistakes.

In concluding this preface we have to stress our special thanks to those who have tirelessly supported us in our endeavour to bring this work to a succesful finish. We thank, in particular, Dr. Niels Jacob, Erlangen, for his assistance in proof reading. Special thanks are due to Professor Heinz Bauer without whose participation it would have been impossible to carry out our task.

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C O N T E N T

INTRODUCTION

Mathematical analysis has undergone its most outstanding and rapid development in the present century. In many instances the classical penetrating results stemming from the beginning of the century have been newly formulated only recently. Certain new methods and specific approaches have brought more understanding of some classical parts and have demonstrated similarities in disciplines where no connections may be observed prima faciae. The fine topologies discussed in this book serve as an example.

The concept of approximate continuity which plays an important part in the theory of real functions was defined at the beginning of this century. But the concept of the density topology naturally connected with that of approximate continuity has been introduced and investigated (relatively) only recently. The fine topology introduced in the forties plays an important part in potential theory. Its methods in the theory of harmonic functions facilitated many excelent results in this classical part of mathematical analysis.

It has become apparent that in spite of having different structures these topologies have many fundamental properties in common. Besides, certain methods and procedures used in connection with density topology have recently influenced the investigation of fine topology in potential theory.

Both the density topology and the fine topology in potential theory are "fine topologies" on a set which has already been equipped with an original coarser (metrizable) topology. Phenomena where two topologies are defined on a set can be encountered in mathematics quite easily. There is even a discipline of general topology - - theory of "bitopological spaces" - which investigates these situations.

Thus nearly the whole first part of our book, i.e. "Abstract Fine Topology", can be considered as belonging to this part of topology. But we investigate mainly those concepts which are related in a way to two fundamental examples of fine topologies and concentrate chiefly upon that part of general theory which is suitable for applications in the theory of real functions or in the potential theory.

Among the general properties of fine topologies one of the most important qualities and also one of the most suitable qualities from the point of applications is the Lusin-Menchoff property to which we devote greater part of this volume. Though it often happens that fine topologies in which we take interest are not normal, nevertheless, they allow us to separate each closed set and finely closed set by "alternately" open sets. We call this property for which the theory of bitopological spaces uses the word "binormality" the Lusin-Menchoff property as it is usual in the theory of real functions when the density topology is investigated. Among others, the Lusin-Menchoff property enables - as well as normality does - various constructions of finely continuous functions.

The fundamental technical means for constructing functions is the "Abstract inbetween theorem" which is simple and not new but which is - perhaps for the first time - explicitly formulated here and provided with a simple proof. Other important concepts are M-modification of a fine topology - which enables to extend finely continuous functions "with the preservation of continuity and semicontinuity" - and the Lusin-Menchoff property of base operators which finds applications in the following text. New "Approximation Theorem" in I/3.6 and its applications also deserve attention.

One of the other problems investigated in this treatise is the following: under which circumstances a space equipped with a fine topology is a Baire space, a strong Baire space or a Blumberg space (I/4), and when finely continuous functions are in Baire class one (I/2D). We also investigate properties of fine limits in an abstract case (I/2C).

Other interesting and not unimportant results (some of which are new) are contained in I/5 which concerns the connectedness of fine topologies.

The beginning of the first chapter is devoted to the study of base operator spaces. But the idea of this generalization of topological spaces is not new at any rate. Among others, we produce some results already proved in the potential theory for concrete base operators in a completely general version. Besides, the concept of base operator (with the Lusin-Menchoff property) appears useful also in real analysis.

In the second part, i.e. in "Fine Topology Methods in Real Analysis", we investigate in detail some fine topologies that occur in real analysis and employ in high degree the general theorems produced in the first part.

The fundamental example of fine topology is a density topology which we study in various degrees of generality (from the density topology on the real line to the "abstract density topologies "which are investigated in lifting theory). We have taken pains to find simple criteria which make it possible to obtain information about some properties of the density topology under the assumption that we know certain properties of the respective "differentiation basis". In our opinion it is especially interesting (and perhaps new) to find simple conditions under which the density topology has the Lusin-Menchoff property (6.34.B), the internal characterization of abstract density topologies (Th. 6.39) and the possibility to use this characterization in lifting theory.

We investigate (II/7.A.7.B) the concepts of a.e.-modification and r-modification of a given fine topology which are generalizations of topologies defined by R.J. O'Malley. The main new contribution is that our proofs enable us to prove the Lusin-Menchoff property of a.e.-topology and r-topology under very general circumstances. The new theorem about the approximation of r-continuous function by approximately differentiable functions is also worth attention.

Another example of a fine topology with the Lusin-Menchoff property is the "fine boundary topology" (II/7.D) and the contingent topology which is related to it and investigated in "Exercises".

The Lusin-Menchoff property of topologies mentioned above is applied in II/8 to obtain results concerning constructions of various functions, especially approximately continuous functions where many of the results are new. Concerning the theorem about the extension from F_σ-sets in II/8.C we would like to mention that we do not use topological methods here.

In II/9 we concentrate upon the boundary behaviour of functions. The chief new idea is use of the Lusin-Menchoff property of "fine boundary topology". The utility of this method is demonstrated in II/9.C,D,E.

The final third part, i.e. "Fine Topology Methods in Potential Theory", is devoted to fine potential theory. The investigation of fine topology and finely hyperharmonic functions in the framework of abstract harmonic spaces (or even more general standard H-cones) culminates in the study of the Dirichlet problem on finely open sets.

Among the purely topological properties of harmonic spaces the Lusin-Menchoff property of the fine topology is of major importance. It is applied in various connections. In chapters 12-14 we present a selfcontained theory of finely hyperharmonic functions and of fine Dirichlet problem in harmonic spaces; until now this theory has been established only under the assumption of the domination axiom D. The possibility of generalization of the presented results to standard H-cones is suggested in the Appendix.

Developing the theory of finely hyperharmonic functions we derive their basic properties (III/12.A) and take interest in the properties of the cone of all nonnegative finely hyperharmonic functions (III/12.B). Besides, we also study the class of all pointwise hyperharmonic functions on a finely open set and its subclass of all its elements that are lower semicontinuous functions. Among new results let us mention e.g. interesting theorem on removable singularities (Th. 12.20, 14.11).

The Dirichlet problem is solved either by means of Perron type method using fine superfunctions or of the Wiener type method when the given finely open set is exhausted by finely regular sets.

A quite new approach to the solution of the fine Dirichlet problem based on quasitopological concepts is shown in Chapter 14. The quasitopological methods are also used to characterize finely hyperharmonic functions (Th. 14.8). The results contained in Chapter 14 embrace both a part of the preceding chapters as well as the basic results of Fuglede's theory of finely hyperharmonic functions in spaces with axiom D.

In the Appendix we investigate analogous problems within the framework of the abstract theory of standard H-cones. We present an abstract definition of the class of "finely hyperharmonic functions" and demonstrate its connections with the localized cone. Our results make it possible to develop the theory of fine Dirichlet problem also in this very general context.

Note that each section ends with "Exercises" containing not only counterexamples and generalizations of the given theorems but often also further new results (even more general structures are investigated). Some of the results of the theory of bitopological spaces, σ-topologies, even further generalizations of abstract density topologies, the study of Keldych operators on finely open sets, a boundary behaviour of the Perron solution etc. may serve as an example. In the "Exercises" one can also find new results and results of other authors having a close relationship to the investigated problems.

The "Exercises" are followed by "Remarks and Comments" where the authors try to give information about the origin of concepts and results of the respective sections and bring occasionally further quotations of authors dealing with similar problems.