Lecture Notes in Mathematics

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Real Functions

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PREFACE

Our intention in this monograph is to survey a number of topics related to the study of the continuity and differentiation properties of real functions, in certain generalized senses. There is now a relatively large literature devoted to such subtle concerns but which is accessible and known only to specialists. Since the ideas are essentially simple and the techniques required fairly elementary this literature should be easily absorbed by any interested mathematician, and it is hoped that the presentation here is sufficiently readable and the exposition adequately clear for this purpose.

Probably the reader needs only a familiarity with the usual basics of real analysis (measure, category, density, etc.) in order to follow the arguments. This material is readily available in a variety of textbooks. A better preparation would be to master the books

S.Saks, Theory of the integral,

and

A.M.Bruckner, Differentiation of real functions,

(references [33] and [209] in the bibliography) that most analysts who work in this particular set of topics would surely consider fundamental to our subject. The present monograph continues certain concerns that arise in each of these works.

Part of this material was presented in a series of seminars at the University of California at Santa Barbara in the spring of 1984, during the special year in Real Analysis that was held there. I am particularly grateful and certainly indebted to the participants in that seminar who offered much helpful criticism and indicated numerous improvements. What remains is, doubtless, flawed but much less so than it would have been without the opportunity to meet with so many fine analysts.

In the first chapter is presented a general structure (called here a local system of sets) that can be used to formulate a variety of general notions of limit, continuity, derivative, etc. for real functions. The reason we have chosen this abstract framework is to enable us to clarify and codify the type of arguments that appear in the study. The greater generality itself is not of much interest; the real intention is to lay bare the underlying analysis. Thus it will appear that almost all of the arguments used in the subject reduce to a few general themes, most notably "intersection conditions" and various thickness conditions (porosity and density usually). All the basic concepts and arguments to be used in the rest of the work are introduced in the first chapter.

The second chapter is a review of the classical material on real cluster sets, from the perspective established in the first chapter. Again the basic arguments here, and elsewhere, will involve appropriate intersection conditions. This cluster set material is attractive and elementary, but does not seem to have been presented in any text to date leaving the interested reader to search through a large number of early references. By restricting ourselves to real cluster sets (i.e. cluster sets for functions of one real variable) we can present an apparently complete survey of the known results.

Chapter three contains a brief account of some general notions of continuity for real functions.

Chapter four gives an introduction to the notion of total variation for a real function. This presentation allows us to include some very classical material on functions of bounded variation, VBG* functions, singular functions, Lebesgue-Stieltjes measures, etc. from a perspective that is not well known and which allows a unified and simple treatment of some apparently diverse ideas.

In Chapter five we have given an account of several classes of monotonicity theorem. This should perhaps be read in conjunction with a study of Chapter XI of Bruckner's monograph ([33,pp.173-198]).

Chapters six and seven are devoted to a number of questions whose theme is the relationship that must hold among different types of generalized derivatives. This includes the well-known Denjoy-Young-Saks theorem and a variety of lesser known variants, both classical and recent.

Finally an Appendix is included that contains a survey on the notion of set porosity. This material too is not well known and can be found, so far, only scattered in the literature. As porosity computations and language appear in many instances in real analysis, this material should be of some use either as a point of reference or as an introduction to the concepts.

There are many more topics that could have been included and which would fit naturally within the framework that we are using. The properties of derivatives and extreme derivates in a generalized sense are currently being studied by some researchers. However this appears still to be in the early stages of development and we have chosen not to report on it. The interested reader should consult the article Bruckner, O'Malley and Thomson [43].

The bibilography contains many articles related to our concerns here, even if not explicitly discussed. It should not be considered complete, however, and the authors whose works I have not mentioned will forgive my oversight.

The notation used is mostly standard nowadays. Thus $A \cup B$, $A \cap B$, and $A \setminus B$ denote the usual union, intersection and difference of the sets A and B, while $\mathbb{R}$ denotes the set of real numbers and $\overline{A}$ the closure of the set A in $\mathbb{R}$. However the peculiarities of the word processor have led to the somewhat old fashioned notation

$$\sum_{k=m}^{n} A_k \quad \text{and} \quad \prod_{k=m}^{n} A_k$$

for the union and intersection of a sequence of sets $\{A_k\}$. This notation should present no difficulties. Other special notations are explained in the text and may be found in the index.

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