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Edited by A. Dold and B. Eckmann

1000

Heinz Hopf

Differential Geometry in the Large

Seminar Lectures New York University 1946
and Stanford University 1956

With a Preface by S.S. Chern

Second Edition

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The editors are happy to make the famous 1946 and 1956 seminar lectures of Heinz Hopf on Geometry and Differential Geometry in the large available to the mathematical community. They are pleased to have this fine volume carry the number 1000 of the Lecture Notes in Mathematics series. They express their sincere thanks to all those who have contributed to the project: To Peter Lax and John Gray who wrote the original class notes; to the Mathematics Institutes of N.Y.U. and of Stanford University for the permission to rewrite and publish the notes; to S.S. Chern for suggesting the volume and writing a preface; to Konrad Voss and Karl Weber for carefully checking the old versions and correcting errors, partly using error lists made by Heinz Hopf himself; and to Rachel Boller for her excellent job in typing the final manuscript and drawing all illustrations.

Albrecht Dold
Beno Eckmann

PREFACE TO THE SECOND EDITION

The text of the Hopf Lecture Notes remains nearly unchanged. A number of misprints has been corrected, for which considerable help was given by WU TA-JEN of Nankai University at Tianjin, China, who also contributed a great number of valuable remarks.

One of the main questions discussed in Part-Two of the Hopf Lectures is the problem of finding all closed surfaces in E^3 with constant mean curvature (c.m.c.), the solution being given in these Lecture Notes for the genus 0 case and for the case of all simple closed surfaces of arbitrary genus (in which cases the round spheres are the only solutions), while "the question whether there exist closed surfaces of genus $\geq$ 1 with H=C and with self intersections ... remains unanswered" (p. 131). An exciting development began in 1986 with H.C. WENTE's proof of the existence of c.m.c. tori; this proof starts exactly at the point, where Heinz Hopf left the problem in 1950. In the meantime, not only have the c.m.c. tori been classified, but N. KAPOULEAS (1987) has also proved the existence of c.m.c. surfaces of arbitrary genus $\geq$ 3. The case of genus 2 still seems to make siffi- culties. For references see the paper of U. PINKALL and I. STERLING: On the classification of constant mean curvature tori, to appear in Annals of Mathematics (1989).

K. Voss
March 1989

PREFACE

These notes consist of two parts:

1) Selected Topics in Geometry, New York University 1946, Notes by Peter Lax.

2) Lectures on Differential Geometry in the Large, Stanford University 1956, Notes by J.W. Gray.

They are reproduced here with no essential change.

Heinz Hopf was a mathematician who recognized important mathematical ideas and new mathematical phenomena through special cases. In the simplest background the central idea or the difficulty of a problem usually becomes crystal clear. Doing geometry in this fashion is a joy. Hopf's great insight allows this approach to lead to serious mathematics, for most of the topics in these notes have become the starting-points of important further developments. I will try to mention a few.

It is clear from these notes that Hopf laid the emphasis on polyhedral differential geometry. Most of the results in smooth differential geometry have polyhedral counterparts, whose understanding is both important and challenging. Among recent works I wish to mention those of Robert Connelly on rigidity, which is very much in the spirit of these notes (cf. R. Connelly, Conjectures and open questions in rigidity, Proceedings of International Congress of Mathematicians, Helsinki 1978, vol. 1, 407-414).

A theory of area and volume of rectilinear polyhedra based on decompositions originated with Bolyai and Gauss. Gauss realized the delicacy of the problem for volumes, and Hilbert proposed in his famous "Mathematical Problems" that of "constructing two tetrahedra of equal bases and equal altitudes which can in no way be split into congruent tetrahedra..." (Problem no. 3). This was immediately solved by Max Dehn whose results, with some modifications, are presented in Part 1, Chapter IV of these notes. This work has been further pursued and treated by algebraic methods. For the modern developments I refer to C.H. Sah, Hilbert's third problem: Scissors congruence (Research Notes in Mathematics 33, Pitman, San Francisco 1979).

The main content of Part 2 consists of the study of Weingarten surfaces in the three-dimensional Euclidean space, particularly those for which the mean curvature or the Gaussian curvature is a constant. Important progress was recently made by Wu-Yi Hsiang, as he constructed many examples of hypersurfaces of constant mean curvature in the Euclidean space which are not hyperspheres; cf. Wu-Yi Hsiang, Generalized rotation hypersurfaces of constant mean curvature in the Euclidean spaces I (J. Differential Geometry 17 (1982), 337-356), and his other papers. But the simplest question as to whether there exists an immersed torus in the three-dimensional Euclidean space with constant mean curvature remains unanswered (the "soap bubble" problem).

Hopf's mathematical exposition is a model of precision and clarity. His style is recognizable in these notes.

S.S. Chern
March 1983

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