Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

804

Michihiko Matsuda

First Order Algebraic Differential Equations

A Differential Algebraic Approach

Springer-Verlag
Berlin Heidelberg New York 1980

Author

Michihiko Matsuda
Department of Mathematics, Kyoto Sangyo University
Kamigamo, Kyoto 603/Japan

AMS Subject Classifications (1980): 12H05

ISBN 3-540-09997-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-09997-2 Springer-Verlag New York Heidelberg Berlin

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2141/3140-543210

Introduction

The study of first order algebraic differential equations produced fruitful results around the end of the last century. The classification of equations free of movable singularities was carried out successfully. The investigations were carried out in the complex plane and the main tool of investigation was "analytic continuation". Fuchs tried to clarify the algebraic aspect making use of "Puiseux series", but his work was not developed fully at that time.

The modern theory of differential algebra and algebraic function fields of one variable has enabled us to give an abstract treatment, leaving the complex plane. Recently the author presented a differential-algebraic criterion for a first order algebraic differential equation to have no movable singularity suggested by Fuchs' criterion for this property. From this standpoint we reconstructed some classical theorems due to Briot, Bouquet, Fuchs and Poincaré. In this treatment the coefficient field is an arbitrary algebraically-closed differential field of characteristic 0.

E. R. Kolchin , using Galois theory of differential fields, obtained in 1953 a theorem containing a criterion for a first order algebraic differential equation to define elliptic functions (cf. §12). The author would like to note that his work was motivated by this excellent theorem. M. Rosenlicht applied valuation theory to the problem of explicit solvability of certain algebraic differential equations successfully.

In this note we shall consider the general case in which the coefficient field is an arbitrary differential field: It is not necessarily of characteristic 0 nor algebraically closed. We assume the reader to be familiar with the contents of the first six chapters of the book "Introduction to the theory of algebraic functions of one variable" by Chevalley (Amer. Math. Soc. 4th printing, 1971), which will be referenced as [C]. Any theorem not contained in this book and used here will be proved, even if the proof is well known. A familiarity with differential algebra is not assumed except in §18.

In §§16-17 recent results of Keiji Nishioka will be introduced: They are valid only in the case of characteristic 0.

The author would like to express his sincere gratitude to Professor I. Laine and Professor M. Rosenlicht for their invitation to the Colloquium on Complex Analysis, Joensuu, Finland, August 24-27, 1978 and the Special Lecture at the University of California, Berkeley, July 18, 1978 respectively.

August 1979

Michihiko Matsuda

Preliminaries

Let M be a commutative ring and N be a subring of M. A derivation of N into M is a mapping δ which satisfies the following conditions:

$$\delta(y + z) = \delta y + \delta z, \quad \delta(yz) = y\delta z + z\delta y.$$

A derivation of M into itself is called simply a derivation of M: If M has no proper zero divisors, then it can be extended to a derivation of the quotient field of M in one and only one way.

Suppose that M is a field and N is a subfield of M. The derivation δ of N into M satisfies

$$\delta(y/z) = (z\,\delta y - y\delta z)/z^2, \quad z \neq 0.$$

If two derivations of N into M coincide on some subset E of N, then they coincide on the subfield of N generated by the elements of E.

Let x be an element of the field M. If x is transcendental over the subfield N, the derivation δ can be extended to a derivation D of N(x) into M such that Dx is an arbitrarily chosen element of M(cf. §1, p.3). Suppose that x is algebraic over N. If x is separable over N, δ can be extended to a derivation of N(x) into M in one and only one way. If an element u of N is the p-th power of x, δ can be extended to a derivation D of N(x) into M if and only if $\delta u = 0$, where p is the characteristic of N. In this case an arbitrary element of M may be taken for Dx if $x \notin N$(cf. for instance pp.12-13 of the book "Foundations of algebraic geometry" by Weil(Amer. Math. Soc., 2nd Edition, 1962)).

Table of Contents

§1. Differential algebraic function field of one variable.... 1

§2. Movable singularities.................................... 7

§3. Fuchs' theorem... 11

§4. Extensions of the coefficient field...................... 17

§5. Briot-Bouquet's theorem.................................. 22

§6. Clairaut fields.. 27

§7. Poincaré fields.. 33

§8. Analogs of addition formulae for elliptic functions...... 42

§9. Poincaré fields generated by the constants............... 47

§10. Poincaré fields of characteristic 0..................... 52

§11. Automorphisms of an elliptic field...................... 56

§12. Differential automorphisms.............................. 69

§13. Differential automorphisms of a Poincaré field.......... 74

§14. Riccati fields of characteristic 0...................... 78

§15. Weierstrass points...................................... 83

§16. Clairaut fields of characteristic 0..................... 91

§17. Movable singularities in case of characteristic 0....... 95

§18. Differential Lefschetz principle........................ 103

Bibliography... 110

Index.. 111