

# Lecture Notes in Mathematics

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Klaus Floret

## Weakly Compact Sets

Lectures Held at S.U.N.Y., Buffalo,  
in Spring 1978

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Weakly compact subsets, i.e. sets which are compact with respect to the weak topology of a Banach-space or more generally: of a locally convex space play an important rôle in many questions of analysis. Among them are characterizations of reflexivity, characterizations of subsets with elements of least distance in linear and convex approximation theory, ranges of vector measures and existence theorems in optimal control theory, pointwise convergence of sequences of functions, minimax-theorems, separation properties of convex sets.

The intention of these lecture notes is to prove the main results on weak compactness due to W.F. Eberlein, V.L. Šmulian, M. Krein, A. Grothendieck, and R.C. James as well as to go into some of the questions mentioned above. There are three foci: the theorems on countable compactness, on sequential compactness, and the supremum of linear functionals. The linking element is A. Grothendieck's interchangeable double-limit property. The results on countable and sequential compactness are, as usual, first proved in spaces of continuous functions, equipped with the topology of pointwise convergence. The approach to R.C. James' theorem and its various applications is the original one in the form which was given by J.D. Pryce: His proof is just checked carefully and the result stated as a double-limit-theorem which implies many of the applications of other versions due to S. Simons and M. DeWilde. A short look into the contents shows that emphasis is put on R.C. James' theorem. A reader who is just interested in this, may start with §5 provided she or he accepts the W.F. Eberlein-A. Grothendieck-theorem (1.6.), the W.F. Eberlein-V.L. Šmulian-theorem (3.10.) and a consequence of

it, A. Grothendieck's theorem on weak compactness in  $C(K)$  (4.2. and 4.4.).

The typical reader whom I have in mind knows the basic facts on locally convex spaces and became somehow interested in weakly compact sets: either by some applications, or by their rôle in the general theory, some properties of them, or from a topological point of view. Consequently, the introductory remarks on locally convex spaces do not serve the purpose of explaining what locally convex spaces are and what they are for; they simply try to go through those parts of the theory which will be needed later on - with the additional benefit that some more or less standard notation will be fixed. Relative to these facts and some knowledge of topology (e.g. a compact space is a Hausdorff topological space such that every net has a cluster-point) the exposition is self-contained.

There are exercises attached to each section: I simply believe that it is much easier to understand a result once one has solved a related problem. At first glance, some of the exercises may seem to be difficult, but on the basis of the foregoing text and the hints the reader should be able to master them. I do not claim that a result stated as an exercise is easy in an "absolute" sense: I only say that at this point of the text there is enough information available to prove it without too much extra effort.

The notes are based on lectures I gave at the State University of New York at Buffalo during the Spring semester of 1978. They may serve as well as a basis for a seminar.

While preparing these lectures I was deeply influenced by the seminar-notes [7] of M. De Wilde and two papers of J.D. Pryce, one [40] presenting a smoothed proof of R.C. James' theorem, the other one [41] dealing with H.D. Fremlin's notion of an "angelic" space.

I thank the Department of Mathematics of S.U.N.Y.A.B. for the invitation to spend the academic year 1977/78 in Buffalo and the colleagues and friends there who created a kind and open atmosphere for me. P. Dierolf, W. Govaerts, M. Wriedt, and V. Wrobel made many valuable remarks on the text. Special thanks to Mrs. Marie Daniel who typed the manuscript with great patience and diligence. I am grateful to the editors for accepting these notes for publication in the Lecture Notes Series.

October 1978

Klaus Floret

CONTENTS

§0 Some fundamentals of locally convex spaces..... 1

0.2. Weak topology, dual systems; 0.3. Mackey-topology;  
0.4. Barrels, barrel-lemma, bounded sets; 0.5. Strong topology,  
semi-reflexive spaces; 0.6. Grothendieck's completeness  
criterion; 0.7. Extreme points; 0.8. (LF)-spaces.

§1 Countably compact sets and the theorem of Eberlein-Grothendieck.... 7

1.1. Definition of countably and sequentially compact sets;  
1.2. Basic properties and counter-examples; 1.4. The inter-  
changeable double-limit-property, pointwise convergence and  
relatively compact sets in  $C(X,Z)$ ; 1.5. Pointwise compact-  
ness in  $C(X)$  and  $C^b(X)$ ; 1.6. Weak countable compactness in  
locally convex spaces: Eberlein-Grothendieck theorem;  
1.8. Other locally convex topologies  
Exercises: 1.15. A criterion of V. L. Šmulian; 1.22. Another  
approach to the Eberlein-Grothendieck theorem.

§2 Bounding sets in the weak topology..... 21

2.1. Bounding and pseudocompact sets, the Tychonoff-Plank;  
2.3. Weakly bounding = weakly relatively compact; 2.5. Weakly  
pseudocompact = weakly relatively compact; 2.7. Other locally  
convex topologies.  
Exercises

§3 Sequential compactness and angelic spaces..... 28

3.1. The angelic-lemma; 3.2. Šmulian's theorem for locally  
convex spaces with weakly separable dual; 3.3. Angelic spaces,  
the basic theorem; 3.5. Fremlin's result; 3.6. Some sets with  
closure = sequential closure (DeWilde); 3.7.  $C(X,Z)$  being  
pointwise angelic; 3.8. and 3.9. The Kaplansky result on  
closures; 3.10. Weakly angelic locally convex spaces: Eberlein-  
Šmulian theorem  
Exercises: 3.17. Products of angelic spaces; 3.20. Another  
approach to pointwise angelic spaces  $C(X,Z)$ ; 3.27. Weakly  
integrable, vector-valued functions.

- §4 Pointwise and weak compactness in spaces of continuous functions.. 45
- 4.1. Compact-open and bounding-open topology; 4.2. Compactness in  $C(K)$  - Grothendieck's theorem; 4.3. In  $C_{CO}(X)$ ; 4.4. In  $C^b(X)$ ; 4.5. The repletion; 4.7. Bounding sets in  $\cup X$ ; 4.8. In  $C_{bdg}(X)$ ; 4.9. Convex sets.  
Exercises: 4.11. Locally compact  $X$ ; 4.24. Measurable functions; 4.25. Cauchy-sequences.
- §5 Best approximations and the theorem of R.C. James..... 57
- 5.1. Mazur's observation; 5.2. Best-approximation; 5.3. The evolution to James' theorem; 5.4. The reflexivity-criterion; 5.5. Sequences of convex sets (Dieudonné-Šmulian theorem) and proximal sets; 5.6. James' theorem does not hold in normed spaces.  
Exercises: 5.17. and 5.19. More characterisations of weakly compact sets.
- §6 Proof of the theorem of R.C. James..... 67
- 6.2. Sketch of the proof; 6.4. Pryce's result on bounded sequences in  $\ell^\infty$ ; 6.5. Sublinear functionals; 6.6. James' double limit-theorem; 6.7. The double-limit inequality; 6.9. and 6.10. Sets with interchangeable double-limits in  $\ell^\infty$ ; 6.11. Attaining the supremum on a subset.  
Exercises: 6.20. Pointwise convergence in  $C(X)$ ,  $X$  pseudo-compact: Simons' result.
- §7 Applications of the sup-theorem..... 82
- 7.1. Krein's theorem on the convex hull of compact sets; 7.3. and 7.4. Closed sums of convex sets; 7.5. On the unit ball of Banach-spaces; 7.6. The convex hull of two convex sets; 7.7. Separation of convex sets; 7.8. Range of vector-measures; 7.9. Fixed points; 7.10. Peano's theorem in non-reflexive Banach-spaces.  
Exercises: 7.13. Interchangeable double-limits of the convex hull; 7.30. Representation of weakly compact operators; 7.31. Uniformly convex spaces.
- §8 The topology related to Rainwater's theorem..... 98
- 8.1. Rainwater's theorem in Choquet-theory and Tweddle's idea; 8.2. The weak topology coming from extreme points of equi-continuous sets; 8.3. Compactness results; 8.4. Pointwise

convergence; 8.5. Weak convergence in  $L^1$ ; 8.6. Uniformly integrable sets; 8.7. Schur's lemma; 8.9. Dunford-Pettis' characterisation of weak compactness in  $L^1$ ; 8.10. - 8.12.  $\epsilon$ -tensor products; 8.13. Vector-valued continuous and differentiable functions.

Exercises: 8.23. Measures with densities; 8.24. Convergence in measure.

|                                 |     |
|---------------------------------|-----|
| Bibliography.....               | 117 |
| References to the sections..... | 120 |
| List of symbols and spaces..... | 121 |
| Index.....                      | 122 |