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Weakly Compact Sets

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Weakly compact subsets, i.e. sets which are compact with respect to the weak topology of a Banach-space or more generally: of a locally convex space play an important rôle in many questions of analysis. Among them are characterizations of reflexivity, characterizations of subsets with elements of least distance in linear and convex approximation theory, ranges of vector measures and existence theorems in optimal control theory, pointwise convergence of sequences of functions, minimax-theorems, separation properties of convex sets.

The intention of these lecture notes is to prove the main results on weak compactness due to W.F. Eberlein, V.L. Šmulian, M. Krein, A. Grothendieck, and R.C. James as well as to go into some of the questions mentioned above. There are three foci: the theorems on countable compactness, on sequential compactness, and the supremum of linear functionals. The linking element is A. Grothendieck's interchangeable double-limit property. The results on countable and sequential compactness are, as usual, first proved in spaces of continuous functions, equipped with the topology of pointwise convergence. The approach to R.C. James' theorem and its various applications is the original one in the form which was given by J.D. Pryce: His proof is just checked carefully and the result stated as a double-limit-theorem which implies many of the applications of other versions due to S. Simons and M. DeWilde. A short look into the contents shows that emphasis is put on R.C. James' theorem . A reader who is just interested in this, may start with §5 provided she or he accepts the W.F. Eberlein-A. Grothendieck-theorem (1.6.), the W.F. Eberlein-V.L. Šmulian-theorem (3.10.) and a consequence of it, A. Grothendieck's theorem on weak compactness in C(K) (4.2. and 4.4.).

The typical reader whom I have in mind knows the basic facts on locally convex spaces and became somehow interested in weakly compact sets: either by some applications, or by their rôle in the general theory, some properties of them, or from a topological point of view. Consequently, the introductory remarks on locally convex spaces do not serve the purpose of explaining what locally convex spaces are and what they are for; they simply try to go through those parts of the theory which will be needed later on - with the additional benefit that some more or less standard notation will be fixed. Relative to these facts and some knowledge of topology (e.g. a compact space is a Hausdorff topological space such that every net has a cluster-point) the exposition is self-contained.

There are exercises attached to each section: I simply believe that it is much easier to understand a result once one has solved a related problem. At first glance, some of the exercises may seem to be difficult, but on the basis of the foregoing text and the hints the reader should be able to master them. I do not claim that a result stated as an exercise is easy in an "absolute" sense: I only say that at this point of the text there is enough information available to prove it without too much extra effort.

The notes are based on lectures I gave at the State University of New York at Buffalo during the Spring semester of 1978. They may serve as well as a basis for a seminar.

While preparing these lectures I was deeply influenced by the seminar-notes [7] of M. De Wilde and two papers of J.D. Pryce, one [40] presenting a smoothened proof of R.C. James' theorem, the other one [41] dealing with H.D. Fremlin's notion of an "angelic" space.

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