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C. T. Chong

Techniques of Admissible Recursion Theory

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Author

Chi-Tat Chong
Department of Mathematics, National University of Singapore
Kent Ridge, Singapore 0511, Singapore

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FOREWORD

The generalization of recursion theory on the set ω of natural numbers to infinite ordinals (now known as higher recursion theory) has a complicated history. The guiding principle was that basic notions such as computation, finiteness, recursion, recursiveness, relative recursiveness and effectiveness, which lie at the heart of the subject of classical recursion theory, should not be confined to the consideration of ω (and hence countable structures) alone. Indeed in various parts of mathematical logic one had encountered notions and procedures which reminded one of effective computability in the generalized sense. In order to develop a reasonable higher recursion theory, however, it was necessary to have a good intuition of the roles played by objects such as finite sets and r.e. sets in the classical setting. Thus for example in Kreisel [1971] it is aptly pointed out why in any reasonable generalization of recursion theory the notion of 'finiteness' ought to be central. As early as 1938, Kleene had introduced the idea of ordinal notations for natural numbers. This made it possible to discuss certain countably infinite ordinals (the <u>recursive ordinals</u>) in terms of the natural numbers they denote. In the 1950's and early 1960's various recursion-theoretic results on Church-Kleene ω_1 (ω_1^{CK}), the least non-recursive ordinal, were derived, leading to the belief that a recursion theory on this ordinal (later christened 'metarecursion theory' by Kreisel and Sacks) could be developed modelled after classical recursion theory. In particular there should be a correct analog of relative recursiveness which would allow a positive solution to Post's problem for ω_1^{CK}. This was subsequently confirmed in Sacks [1966a], [1966b].

It turns out that Gödel's constructible universe L is an ideal structure in which to develop higher recursion theory. In retrospect one could view the eight fundamental operations introduced by Gödel, in his work on the consistency of the Axiom of Choice and the Generalized Continuum Hypothesis, as recursive functions defined on the universe of sets. Indeed in Takeuti [1960] a study was made on the recursive functions defined on the class of ordinals and it was shown that the cardinals are closed under these functions. Kripke [1964] introduced the notion of admissible ordinals, showing that ω_1^{CK} and all infinite cardinals are instances of these ordinals. He further demonstrated that 'finiteness' in an admissible ordinal could be identified with the property of being an element of L_α (Gödel's L at level α). Closely related to this was Platek's approach [1966] which developed higher recursion theory via definability. He also established the equivalence between basic recursion-theoretic and set-theoretic statements. Recursion theory on admissible ordinals thus evolved into a special case of higher (admissible) recursion theory which could in turn be viewed as a fragment of the Zermelo-Fraenkel set theory. This has since come to be known as the Kripke-Platek set theory. In 1966-67 Sacks introduced the method of priority argument into ordinal recursion theory, solving Post's problem for many admissible ordinals. Finally Jensen's work on the fine structure theory of L ([1972]) provided the necessary tools and insights into admissible recursion theory. Very soon after this, Sacks and Simpson [1972] obtained the complete solution of Post's problem for all admissible ordinals.

This monograph attempts to present an account of the development of the subject of admissible recursion theory that began with the Sacks-Simpson paper. Under the influence of Sacks, higher recursion theory has expanded from admissibility to inadmissibility (Friedman and Sacks [1977]), and into E-recursion (Sacks [1982]). Many of the techniques first introduced to solve problems for admissible ordinals have since been applied to tackle the corresponding ones for inadmissible ordinals. For reason of space, we will devote our attention solely to the admissible case. The emphasis is on introducing the techniques that have been invented to solve problems in the area (hence the title of the book). Each technique is illustrated, as it were, by a result whose proof was the source of the introduction of the particular technique. We give an outline below:

Chapter 1 consists of preliminaries, reviewing some of the basic recursion-theoretic notions and Jensen's fine structure theory. We prove Maass's theorem that (uniformly) there is an α-regular set in every α-r.e. degree. Various characterizations of definable projecta due to Jensen and Simpson are also proved. Chapter 2 discusses forcing over admissible ordinals, and proves Simpson's result on the jump of α-degrees. Chapter 3 introduces the α-finite injury method. We first prove the Sacks-Simpson theorem on the solution of Post's problem, followed by discussing the blocking technique invented by Shore to prove the Splitting Theorem. We introduce the e-state method in Chapter 4, giving Lerman's characterization of α on the existence of maximal sets. This method is studied further in Chapter 5 where we investigate the existence of major subsets (results of Lerman, Leggett-Shore and Chong). In Chapter 6 we study hyperhypersimple sets, and prove existence and non-existence results of these sets (due to Chong and Lerman) on various admissible ordinals. Chapter 7 studies infinite injury method by proving Shore's theorem on the existence of minimal pairs. This study is continued in Chapter 8 where we give a simpler proof of the Density Theorem for all admissible ordinals due originally to Shore. We turn our attention to arbitrary α-degrees in Chapter 9, where we study the existence of minimal degrees and prove the results of Shore and Maass. The notion of the admissible collapse due to Maass is also introduced. In Chapter 10 we turn to set-theoretic methods, where S. Friedman's theorem on the ordering of the degrees above **0'** for many singular cardinals with uncountable cofinality are proved. The book ends with an appendix which presents a proof of Harrington's result that there exists an admissible set for which Post's problem has a negative solution. It is the appropriate place with which to end the treatment since one can say without exaggeration that Post's problem started the whole subject of modern recursion theory.

It should be pointed out that there is an alternative to the approach we have adopted here. Namely, instead of dividing the chapters according to finite injury method, e-state method etc., one could profitably approach it from the point of view of definable projecta. More precisely, various ordinals have been identified to be those where a successful priority argument could be carried out once the set of requirements are indexed within them. For example many finite injury methods can be carried out on requirements of length α^*, the Σ_1 projectum of an admissible ordinal α. The tame Σ_2 projectum turns out to be sufficient to do practically all finite injury arguments. Also Lerman [1974] has shown that the value of the Σ_3 projectum of α is the key to the existence or non-existence of maximal sets. One can argue rather forcefully that this alternate approach brings into sharper focus the contrast between classical and admissible recursion theory, from which the basic features of recursion theory on ω are reflected. Our point of view is somewhat different. The fundamental constructions in α-recursion theory invented thus far consist of those which are discussed in this book. The definable projecta are introduced to ensure that these constructions work. Furthermore recent investigations have concentrated more on specific ordinals where problems have remained unsolved for sometime. As most of these ordinals are constructible cardinals, they are their own definable projecta. This means that the method of coding the set of requirements by a short indexing set fails completely. Nevertheless, some impressive results have surfaced with the use of set-theoretic methods (Friedman [1981]). Some of the most challenging problems in recursion theory today are on the structure of degrees over constructibly singular cardinals (for example the minimal $\aleph_\omega^L$-degree problem). The approach taken here is intended to lay emphasis to this.

We assume that the reader is familiar with classical recursion theory. Lerman [1983], Soare [1984] and Rogers [1967] are recommended for reference material.

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C. T. Chong

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