Lecture Notes in Mathematics 1200

Editors:
J.-M. Morel, Cachan
F. Takens, Groningen
B. Teissier, Paris

Springer
Berlin
Heidelberg
New York
Barcelona
Hong Kong
London
Milan
Paris
Singapore
Tokyo

Vitali D. Milman   Gideon Schechtman

Asymptotic Theory of Finite Dimensional Normed Spaces

With an Appendix by M. Gromov
"Isoperimetric Inequalities in Riemannian Manifolds"

Springer

Authors

Vitali D. Milman
Department of Mathematics
Tel Aviv University
Ramat Aviv, Israel

Gideon Schechtman
Department of Theoretical Mathematics
The Weizmann Institute of Science
Rehovot, Israel

Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Milman, Vitali D.:
Asymptotic theory of finite dimensional normed spaces / Vitali D. Milman ; Gideon Schechtman. With an appendix Isoperimetric inequalities in Riemannian manifolds / by M. Gromov. - Corr. 2. printing. - Berlin ; Heidelberg ; New York ; Barcelona ; Hong Kong ; London ; Milan ; Paris ; Singapore ; Tokyo : Springer, 2001
(Lecture notes in mathematics ; 1200)
ISBN 3-540-16769-2

Corrected Second Printing 2001

Mathematics Subject Classification (1980): 46B20, 52A20, 60F10

ISSN 0075-8434
ISBN 3-540-16769-2 Springer-Verlag Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting, reproduction on microfilms or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

Springer-Verlag Berlin Heidelberg New York
a member of BertelsmannSpringer Science+Business Media GmbH

http://www.springer.de

© Springer-Verlag Berlin Heidelberg 1986
Printed in Germany

Typesetting: Camera-ready TEX output by the authors
SPIN: 10797471 41/3142-543210 - Printed on acid-free paper

INTRODUCTION

This book deals with the geometrical structure of finite dimensional normed spaces, as the dimension grows to infinity. This is a part of what came to be known as the Local Theory of Banach Spaces (this name was derived from the fact that in its first stages, this theory dealt mainly with relating the structure of infinite dimensional Banach spaces to the structure of their lattice of finite dimensional subspaces).

Our purpose in this book is to introduce the reader to some of the results, problems, and mainly methods developed in the Local Theory, in the last few years. This by no means is a complete survey of this wide area. Some of the main topics we do not discuss here are mentioned in the Notes and Remarks section. Several books appeared recently or are going to appear shortly, which cover much of the material not covered in this book. Among these are Pisier's [Pis6] where factorization theorems related to Grothendieck's theorem are extensively discussed, and Tomczak-Jaegermann's [T-J1] where operator ideals and distances between finite dimensional normed spaces are studied in detail. Another related book is Pietch's [Pie].

The first major result of the Local Theory is Dvoretzky's Theorem [Dv] of 1960. Dvoretzky proved that every real normed space of finite dimension, say n, contains a $(1+\varepsilon)$-isomorphic copy of the k-dimensional euclidean space ℓ_2^k, for $k = k(\varepsilon, n)$ which increases to ∞ with n (see Chapter 5 for the precise statement). Dvoretzky's original proof was very complicated and understood only by a few people. In 1970 Milman [M1] gave a different proof which exploited a certain property of the Haar measure on high dimensional homogeneous spaces, a property which is now called the concentration phenomenon: Let (X, ρ, μ) be a compact metric space (X, ρ) with a Borel probability measure μ. The concentration function $\alpha(X, \varepsilon)$, $\varepsilon > 0$, is defined by

$$\alpha(X,\varepsilon) = 1 - inf\{\mu(A_\varepsilon);\ \mu(A) \geq \frac{1}{2},\ A \subseteq X\ Borel\}$$

where

$$A_\varepsilon = \{x \in X;\ \rho(x, A) \leq \varepsilon\}\ .$$

It turns out that for some very natural families of spaces, $\alpha(X, \varepsilon)$ is extremely small. For example, it follows from Levy's isoperimetric inequality that for the Euclidean n-sphere S^n, with the geodesic distance ρ and the normalized rotational invariant measure μ,

$$\alpha(S^n, \varepsilon) \leq \sqrt{\frac{\pi}{8}} exp(-\varepsilon^2 n/2)\ .$$

It follows from this inequality (see Chapter 2) that any nice real function on S^n must be very close to being a constant on all but a very small set (the exceptional set being of measure of order smaller than $exp(-\varepsilon^2 n/2)$). This last property is what is called the concentration phenomenon. It has proved to be extremely useful in the study of finite dimensional normed spaces.

Going back to the concentration function, we define a family (X_n, ρ_n, μ_n) of metric probability spaces to be a Levy family if $\alpha(X_n, \varepsilon_n \, diam \, X_n) \underset{n\to\infty}{\longrightarrow} 0$ ($diam \, X_n$ is the diameter of X_n). Chapter 6 below contains a lot of examples of such natural families. Many of these examples have deep applications in the Local Theory. It is usually quite a difficult task to establish that a certain family is a Levy family, the methods are different from one example to the other and come from diverse areas (including methods from differential geometry, estimation of eigenvalues of the Laplacian, large deviation inequalities for martingales, isoperimetric inequalities). Levy families, the concentration phenomenon and their applications to the asymptotic theory of normed spaces are the main topics of the first part of this book. We have already mentioned one application, namely Dvoretzky's Theorem. In the same direction we deal with estimation of the dimension of euclidean subspaces of various large families of normed spaces, that is, with the evaluation of the function $k(\varepsilon, n)$ mentioned above when restricted to some (wide) families of normed spaces. (This study originated in [M1] and [F.L.M.].) Here is an example: There exists a function $c(\varepsilon) > 0$, $\varepsilon > 0$, such that if $X_n = (\mathbb{R}^n, \|\cdot\|)$ is a family of normed spaces, then either for some $\alpha > 0$ and any $\varepsilon > 0$ and n, X_n contains a $(1+\varepsilon)$-isomorphic copy of ℓ_2^k with $k = [c(\varepsilon)n^\alpha]$, or for any integer k and any $\varepsilon > 0$, there is an n such that X_n contains a $(1+\varepsilon)$-isomorphic copy of ℓ_∞^k. (The proof of this result uses, besides the concentration phenomenon, also the notions of type and cotype introduced below.)

We also deal, in the first part, with packing high dimensional ℓ_p^k, $1 < p < 2$, spaces into ℓ_1^n (Chapter 7) as well as packing spaces with special structure (unconditional or symmetric bases) into general normed spaces (Chapter 10).

The second part of the book revolves around the notions of type and cotype and the relation of these notions to the geometry of normed spaces.

For $1 \le p \le 2 \le q \le \infty$ the type p constant (resp. cotype q constant) of X denoted $T_p(X)$ $(C_q(X))$ is the smallest constant in the inequality

$$\left(\operatorname*{Ave}_{\varepsilon_i=\pm 1} \|\sum_{i=1}^k \varepsilon_i x_i\|^2\right)^{1/2} \le T\left(\sum_{i=1}^k \|x_i\|^p\right)^{1/p}$$

$$\left(\quad \left(\sum_{i=1}^k \|x_i\|^q\right)^{1/q} \le C\left(Ave\|\sum_{i=1}^k \varepsilon_i x_i\|^2\right)^{1/2}\quad\right)$$

for all k and $x_1, \dots, x_k \in X$.

These notions were introduced by Hoffmann-Jorgensen [H-J] for the study of limit theorems for vector valued random variables, and were studied extensively by Maurey and Pisier ([M.P.] in particular) in connection with the geometry of normed spaces. They have proved to be a very important tool in the Local Theory. In particular, Krivine, Maurey and Pisier showed that for $p_X = sup\{p;\ T_p(X) < \infty\}$ (resp. $q_X = inf\{q;\ C_q(X) < \infty\}$), $\ell^n_{p_X}$ (resp. $\ell^n_{q_X}$) are $(1+\varepsilon)$-isomorphic to subspaces of X for all n and $\varepsilon > 0$ (here X is infinite dimensional; there is also a corresponding statement for finite dimensional spaces). We present a proof (somewhat different from previous proofs) of this theorem in Chapters 12 and 13. (Chapter 11 deals with some infinite dimensional combinatorial methods needed in the sequel.)

Chapter 14 is devoted to the work of Pisier [Pis1] estimating the norm of one specific projection called the Rademacher projection. This is closely related to the relation between the type p constant of X and the cotype q constant of X^* $\left(\frac{1}{p}+\frac{1}{q}=1\right)$. It also has applications to finding well complemented euclidean sections in normed spaces. These applications due to Figiel and Tomczak-Jaegermann [F.T.] are discussed in Chapter 15.

The book also contains five appendices, the first of which is written by M. Gromov and gives an introduction to the theory of isoperimetric inequalities on riemannian manifolds. It is written in a way understandable to the non-expert (in Differential Geometry). This appendix contains also results which were not published elsewhere (in particular – the Gromov-Levy isoperimetric inequality). We are indebted to M. Gromov for this excellent addition to our book.

CONTENTS

Appendices