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Galerkin Finite Element Methods for Parabolic Problems

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PREFACE

The purpose of this work is to present, in an essentially self-contained form, a survey of the mathematics of Galerkin finite element methods as applied to parabolic problems. The selection of topics is not meant to be exhaustive, but rather reflects the author's involvement in the field over the past ten years. The goal has been mainly pedagogical, with emphasis on collecting ideas and methods of analysis in simple model situations, rather than on pursuing each approach to its limits. The notes thus summarize recent developments, and the reader is often referred to the literature for more complete results on a given topic. Because the formulation and analysis of Galerkin methods for parabolic problems are generally based on facts concerning the corresponding stationary elliptic problems, the necessary elliptic results are included in the text for completeness.

The following is an outline of the contents of the notes:

In the introductory Chapter 1 we consider the simplest Galerkin finite element method for the standard initial boundary value problem with homogeneous Dirichlet boundary conditions on a bounded domain for the heat equation, using the standard associated weak formulation of the problem and employing first piecewise linear and then more general piecewise polynomial approximating functions vanishing on the boundary of the domain. For this model problem we demonstrate the basic error estimates in energy and mean square norms, first for the semidiscrete problem resulting from discretization in the space variables only and then also for the most commonly used completely discrete schemes obtained by discretizing the semidiscrete equation with respect to the time variable.

In the following five chapters we consider several extensions and generalizations of these results in the case of the semidiscrete approximation, and show error estimates in a variety of norms. First, in Chapter 2, we express the semidiscrete problem by means of an approximate solution operator for the elliptic problem in a manner which does not require the approximating functions to satisfy the homogeneous boundary conditions. A discrete method of Nitsche based on a non-standard weak formulation of the elliptic problem is used as an example. In Chapter 3 more precise results are shown in the case of the homogeneous heat equation. These require an accurate description of the smoothness of the solution for given initial data, expressed in terms of certain function spaces $\dot{H}^s(\Omega)$ which will be used repeatedly in these notes and which take into account both the smoothness and the boundary behavior of its elements. We also demonstrate that the smoothing property of the solution operator for positive time has an analogue in the semidiscrete situation and that, as a consequence, the finite element solution then converges to full order even when the initial data are non-smooth. The results of Chapters 2 and 3 are extended to more general parabolic equations in Chapter 4. In Chapter 5 some a priori bounds and error estimates with respect to the maximum-norm are derived in a simple situation, and in Chapter 6 negative norm estimates are shown, in certain cases together with related results concerning the convergence at specific points (superconvergence).

In the next three chapters we consider the discretization in time of the semidiscrete problem. First, in Chapter 7, we study the homogeneous heat equation and give analogues of our previous results both for smooth and for non-smooth data. The methods used for time discretizations are of one-step type and rely on rational approximations to the exponential, allowing the standard Euler and Crank-Nicolson procedures as special cases. In Chapter 8 we study completely discrete one-step methods for the inhomogeneous heat equation in which the forcing term is evaluated at a fixed finite number of points per time step. In Chapter 9 we apply Galerkin's method for the time discretization and seek discrete solutions as piecewise polynomials in the time variable which may be discontinuous at the nodes of the now not necessarily uniform partition of the time axis. In this procedure the forcing term enters in integrated form rather than at a finite number of points.

In Chapter 10 we discuss the application of the standard Galerkin method to a nonlinear parabolic equation. We show error estimates for the semidiscrete problem and then pay special attention to the formulation and analysis of time stepping procedures which are linear in the unknown functions.

In the following three chapters we consider various modifications of the standard Galerkin method. In Chapter 11 we analyze the so called lumped mass method for which in certain cases a maximum principle is valid. In Chapter 12 we discuss the H^1 and H^{-1} methods. In the first of these, the Galerkin method is based on a weak formulation with respect to an inner product in H^1 and for the second, the method uses trial and test functions from different finite dimensional spaces. In Chapter 13, the approximation scheme is based on a mixed formulation of the initial boundary value problem in which the solution and its gradient are sought independently in different spaces.

In the final Chapter 14 we consider the singular problem obtained by introducing polar coordinates in a spherically symmetric problem in a ball in R^3 and discuss two Galerkin methods based on two different weak formulations defined by two different inner products.

References to the literature are given at the end on each chapter. The numbering of theorems, lemmas and formulas is made for each chapter separately, and when a reference is made to a different chapter this is explicitly stated.

These notes have developed from courses that I have given at the University of Queensland, Australia, in 1979, Université Pierre et Marie Curie (Paris VI) in 1980, and Jilin University, China, in 1982, and also, of course, from my teaching over the years in my own university, Chalmers University of Technology, Göteborg, Sweden. I wish to thank all my students and colleagues in these institutions for the inspiration they have provided. Most of my own work in this field has been intimately connected with my association during more than a decade with J.H. Bramble, A.H. Schatz and L. Wahlbin of Cornell University and I wish to express my gratitude to them for their congenial collaboration and to the U.S. National Science Foundation for supporting this collaboration during 12 summers.

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