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Stability of Unfoldings

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INTRODUCTION

The concept of stability plays a major role in the theory of singularities. There are several reasons for the importance of this notion. For one, usually the problem of classifying the objects being studied is extremely difficult; it becomes much simpler if one tries to classify only the stable objects. For another, in many cases (though not in all) the stable objects are generic, that is, they form an open and dense set; so in these cases almost every object is stable and every object is near to a stable one; the non-stable objects are peculiar exceptions. But a third reason for the importance of stability is that the theory of singularities has in recent years, especially through the ideas of R. Thom, acquired important applications to the natural sciences; stability is a natural condition to place upon mathematical models for processes in nature because the conditions under which such processes take place can never be exactly duplicated; therefore what is observed must be invariant under small perturbations and hence stable.

Stability notions have been defined for a variety of objects occurring in the theory of singularities: for mappings, for map-germs, for varieties, for vector fields, for attractors of vector fields and so on. In some cases very little is known about the stable objects; in some cases the stable objects have been completely classified. Other cases lie between these extremes; for example, characterizations of stable proper smooth mappings between manifolds and of stable smooth map-germs have been given by Mather; he has also computed the dimensions in which the stable proper mappings are dense in the set of all proper mappings (see [7], [8], [9]).

For smooth real valued map-germs the theory is rather trivial; such a germ is stable if and only if it has at worst a non-degenerate singularity (or is non-singular).

In this paper the main topic is an investigation of several notions of stability for unfoldings. If η is a germ at the origin of a smooth real-valued function defined on R^n, then an <u>r-dimensional unfolding</u> of η is a germ $f: R^n \times R^r \longrightarrow R$ of a smooth function defined near the origin, such that $f|R^n \times \{0\}$, considered as a germ on R^n, is η. An r-dimensional unfolding is in effect a germ of a smooth r-parameter family of germs on R^n; the fibration of R^{n+r} as $R^n \times R^r$ plays an important role in the theory of unfoldings; essentially it is part of the structure of an unfolding. For this reason the theory of unfoldings has important applications to Thom's catastrophe theory, since there one considers families of potential functions.

To define a notion of stability, one first needs a notion of equivalence between objects. This is usually given by defining two objects to be equivalent if one can be transformed into the other by homeomorphisms or diffeomorphisms of the underlying space. In the case of unfoldings, these diffeomorphisms will be required to respect the fibration of $R^n \times R^r$; for this reason the theory of stable unfoldings is distinct from the theory of stable germs without additional structure, and in particular is non-trivial.

We define several different notions of stability; most of these are defined geometrically, and the difference between these notions results from choosing different equivalence relations and different topologies on the space of smooth real valued functions defined on an open subset of R^{n+r}. One of the stability notions, infinitesimal stability, is defined by an algebraic condition. The main result of this paper (Theorem 4.11) is that these notions are all equivalent; i.e. there is essentially only one reasonable notion of stability for unfoldings. As a corollary we have an easily verifiable algebraic criterion for stability, since infinitesimal stability is defined by an algebraic condition.

The main application of this result is to state precisely and to prove René Thom's celebrated statement that there are exactly seven "elementary catastrophes". This is our theorem 5.6. Essentially this is a classification theorem for stable unfoldings of unfolding dimension < 4.

In the early part of the paper we investigate some aspects of the theory of germs of smooth real-valued functions and of the theory of unfoldings which are important for the investigation of stability and which also have applications in Thom's catastrophe theory. Our main reference for this part of the paper is a set of notes by John Mather [10] which exist only in manuscript form. These notes have not been published nor does it appear that they are likely to be published in the near future. Therefore we have included in this early part of the paper several of Mather's proofs, since they are not likely to be otherwise available to the reader.

The paper is organized as follows:

§ 1 contains the tools which are needed throughout the rest of the paper. Here we define our notation and recall the definitions of basic concepts. We also quote some major theorems from other sources which will be applied in our investigations, and we prove some easy corollaries of these theorems. The most important results which we cite in this chapter are: Nakayama's lemma, the Malgrange preparation theorem; the Thom transversality lemma; and a lemma of Mather's on constructing certain germs of diffeomorphisms.

§ 2 is concerned with finite determinacy of germs. We work with two equivalence relations between germs: suppose μ and η are germs at 0 of functions on $\mathbb{R}^n$; they are <u>right equivalent</u> if there is a local diffeomorphism φ of $\mathbb{R}^n$ such that $\mu = \eta\varphi$; they are <u>right-left</u> equivalent if there is a local diffeomorphism φ of $\mathbb{R}^n$ and a local diffeomorphism λ of $\mathbb{R}$ such that $\mu = \lambda\eta\varphi$. A germ is right (right-left) k-determined if it is right (right-left) equivalent to any other germ with the same k-jet. We investigate a large number of algebraic criteria for k-determinacy. Many of the results in this section were first proved by Mather [6] and [10]; however the results of [6] are weaker than those we give here and in [10] Mather considers only the "right" case. We have improved slightly upon the results of [6] and generalized the results of [10] to the "right-left" case.

§ 3 deals with unfoldings of germs. Here again there is a "right" and a "right-left" case, depending upon whether we transform unfoldings by mappings on the right only or on the right and the left. In this section we investigate the problem of determining what unfoldings a given germ can have. We give an almost complete solution to this problem by showing that every finitely determined germ has right and right-left universal unfoldings: an unfolding f of a germ η is called right (right-left) universal if it induces every other unfolding of η by composition with mappings on the right or on the right and the left. We also give algebraic characterizations of the universal unfoldings. Here too the results for the "right" case are due to Mather [10]; we have generalized these results to the "right-left" case.

§ 4 is concerned with stability of unfoldings and contains our main result. We define six geometric notions of stability, and one algebraically-defined notion, infinitesimal stability. We prove that infinitesimal stability is equivalent to right-left universality. We conclude the section with our main theorem, that all of the notions of stability we define are equivalent to each other (and hence equivalent to right-left universality).

§ 5 contains the main application of our results, to Thom's catastrophe theory. We give a precise statement of Thom's claim that there are only seven elementary catastrophes, and prove the validity of his list. The results of § 4, which establish a notion of stability for unfoldings and give an algebraic criterion for stability, are necessary for the statement of the Thom theorem and are applied in the proof. A major step in the proof is a classification theorem for germs of low codimension (Lemma 5.15) which is essentially due to Mather [10]; we give the proof of this lemma anyway, since it has not been published. We also give a new proof of the splitting lemma (our Lemma 5.12), which Mather proves in a different way, and with additional hypotheses, in [10]. Gromoll and Meyer prove a generalisation of this lemma in [2].

The paper concludes with a short appendix in which the main ideas of Thom's catastrophe theory are sketched and the relevance of the results of this paper to Thom's theory is explained.

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