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Group Representations

A Survey of Some Current Topics

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To

Irving, Blanche, Lenny and Barry

PREFACE

The idea for these notes was conceived as I watched the progress of several thesis students in my department at the University of Maryland. Although they were all working in representation theory, and although they talked together regularly, each seemed to have a limited idea of what the other was doing. In particular, while one was into semisimple Lie groups, another was involved with nilpotent Lie groups and there was a minimum of mutual understanding. Now, it is true that even the most advanced researchers in these areas have undertaken too little intercommunication; so, it is not at all surprising that students in the field should fare no better.

With an eye towards remedying this unpleasant situation (at least locally), I conducted a survey course in group representations during the spring of 1973. The main goal was to open the students eyes to various different vistas within the general panorama of group representations. A secondary goal, motivated in part by my current work, was to present the various interactions and connections between these fields (as it has rarely if ever been done). A third and final goal was to help the students in their own work by providing numerous examples and exercises, indications of current problems in the field, and copious bibliographical references.

I think the course had a fair degree of success. In any event, several students and faculty have encouraged me to put it in writing to see if anyone else might be interested. The result is these notes. It is my hope that they might prove useful to students trying to learn the field, workers in one area of representation theory who lack familiarity with another, or as a general source of reference.

The topics treated are the following. In Chapter I, I present an introduction to the representation theory of semisimple Lie groups. This is a beautiful and elaborate subject -- due in large part to Harish-Chandra -- and to tell the full story would fill several books (if you don't believe me, ask Garth Warner). I have tried to hit the highlights in a consistent and lively manner. It is my feeling that the current attractiveness of group representations to the mathematical community is due in part to the compelling beauty and power of this particular theory. It is also due to the excellent foundations laid for the theory by G. Mackey a generation ago. That is the subject matter of Chapter II. In order to appreciate Chapter II (and the rest of the book for that matter), the reader needs a good knowledge of induced representations. (In Appendix A, the reader will find a short introduction to induced representations which includes the important basic definitions and properties.) In the many examples that are found in Chapter II (especially in section B), I have tried to present what I feel is one of the few systematic attempts to relate many of Mackey's original results on induced representations to the particular case of semisimple groups.

Chapter III is devoted to Mackey's theory of representations of group extensions. This lovely theory is a natural outgrowth of the material in Chapter II and I have tried to present it that way. Once again, I offer many examples, some of which serve as motivation for the subject matter of Chapter V. I also include in Chapter III the theory (recently developed by Adam Kleppner and myself) of Plancherel measure for group extensions. The Imprimitivity Theorem (discussed in Appendix B) is an indispensable tool in this chapter.

In Chapter IV, I give the story of another splendid success in group representations - the theory of orbits and the representations of simply connected nilpotent Lie groups. The main results are due to Dixmier, Pukanszky and especially Kirillov. In Chapter V, under the general heading of algebraic groups, I tried to do two things. First, I indicated how previously described results on Lie groups go over to p-adic groups (sometimes they don't, unfortunately). The main results are due to Harish-Chandra, Moore and others. Secondly, I tried to suggest to what extent the Mackey theory of group extensions might be applied to interconnect the (heretofore divergent) areas of semisimple and unipotent groups. This work is yet very young, and I feel will receive much attention in the future.

Finally in Chapter VI I give a very brief introduction to the representation theory of solvable groups. For real groups the results are due to Auslander and Kostant, for p-adic groups to Howe.

Although I have covered a broad spectrum of results in representation theory here, I have by no means covered it all. Perhaps the single most important topic omitted is the multiplicity theory of the regular representation of a group G acting on $L_2(G/\Gamma)$, Γ a discrete subgroup. There are other omissions of course as well. Most were dictated by considerations of time. It was only a one-semester course, and I didn't want to get old writing up these notes. So much for errors of omission. As for the other kind, I apologize flat out. If there are any, they are due to laziness, ignorance or my own peculiar way of looking at things.

These notes run very close to the course as originally presented in my lectures. Thus the reader will find many theorems stated without proof, some stated with partial proofs or indications of proof, and a few actually proved in entirety (nobody's perfect). I'm sorry to say that I had no grand scheme for deciding how much proof to supply for any particular theorem -- to a great extent it depended on the day-to-day needs of time as the course progressed. Another word of caution. Sometimes group actions appear on the left, at other times on the right. Don't search for hidden significance -- again convenience dictated the convention.

A word on prerequisites for reading these notes. The reader is expected to have a non-trivial knowledge of representation theory. For example, he should know things that are in the books by Naimark, Loomis, Dixmier (second half of C^*-algebras, anyway), and the introduction to the Auslander-Moore Memoir. In addition, he is assumed to be familiar with Lie groups and Lie algebras, Borel spaces, operator theory, algebraic varieties (a little bit), and a few other things on occasion. Terminology and notation that is not defined in the main text can usually be found in the listing at the back of these notes.

Finally, it is my pleasure to thank Eloise Carlton, Robert Martin, John Pesek, and William Rapley. They were a hardy bunch to have put up with me for fourteen weeks. I am also grateful for the help and encouragement given to me by my colleagues Leon Greenberg and Adam Kleppner, and for the excellent typing job done by Debbie Curran and Betty Vanderslice.

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