

# Lecture Notes in Mathematics

Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

377

---

A. M. Fink

Iowa State University of Science and Technology, Ames, IA/USA

## Almost Periodic Differential Equations

---



Springer-Verlag  
Berlin · Heidelberg · New York 1974

---

AMS Subject Classifications (1970): 34-02, 34-C-25, 42-02,  
42-A-84, 34-C-30, 34-D-20

---

ISBN 3-540-06729-9 Springer-Verlag Berlin · Heidelberg · New York  
ISBN 0-387-06729-9 Springer-Verlag New York · Heidelberg · Berlin

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically those of translation, reprinting, re-use of illustrations, broadcasting, reproduction by photocopying machine or similar means, and storage in data banks.

Under § 54 of the German Copyright Law where copies are made for other than private use, a fee is payable to the publisher, the amount of the fee to be determined by agreement with the publisher.

© by Springer-Verlag Berlin · Heidelberg 1974. Library of Congress  
Catalog Card Number 74-391. Printed in Germany.

Offsetdruck: Julius Beltz, Hemsbach/Bergstr.

## Preface

These lecture notes are written with the hope that the recent advances in the subject of almost periodic differential equations can become more accessible if the basic facts are collected in one place. It is well known that in Celestial Mechanics, almost periodic solutions and stable solutions are intimately related. In the same way, stable electronic circuits exhibit almost periodic behavior. A vast amount of research has been directed toward studying these phenomena. A great portion of the roughly five hundred items in our bibliography are dated after 1955.

These lecture notes are about almost periodic solution to ordinary differential equations. I spend the first four chapters on the theory of almost periodic functions. Included in those chapters is the skeleton of almost periodic theory. I have taken the tack of presenting only that material which is germane to the later chapters on differential equations. I include essentially no fact about almost periodic functions which is not used to prove something else. This is no virtue. It illustrates the depth of the thoery that is developed here.

These notes are self-contained except for the usual preliminary facts about existence and uniqueness of solutions of differential equations. It should therefore be accessible to a wide audience. I have taken the periodic case as motivation for much of the material, so an acquaintance with the fourier series theory of periodic functions is helpful.

Much of the material in the first four chapters appears in other books. Some of the material, however, is only available in research papers. This is primarily the case because the theory of almost periodic differential equations has been a rich source for new developments in almost periodic functions that are not well-known.

I have tried to give references to specific results. The reader will find an extensive bibliography of the subject. In addition, the notes at the end of each chapter are helpful in identifying sources of results as well as sources for a study of the literature.

The table of contents and the index are reasonably extensive and should be useful to the casual reader.

It is obvious that these lecture notes can only be an introduction to the subject. The number of items in the bibliography attests to that. Much interesting material is not mentioned. Topological dynamics is not included. It would require a book in itself, see Jacobs [546]. A specialist in the subject may easily find some favorite idea that is missing. I only hope that many readers can find something to tickle their imagination.

These notes have been used in various forms, for seminars at the University of Nebraska, the University of Colorado, and Iowa State University. The participants in those seminars have materially aided the exposition given here. In particular, my colleague, George Seifert, has read these notes and his comments, when followed, have improved the exposition. I thank him for his efforts.

## Table of Contents

### Preface

Chapter 1: Almost Periodic Functions	1
1. Introduction	1
2. Definition	1
3. The hull	3
4. The space $AP(C)$	5
5. The Bohr definition	7
6. Derivatives of AP functions	10
7. A pointwise definition	11
8. Notes	14
Chapter 2: Uniformly Almost Periodic families	16
1. Introduction	16
2. U.A.P. families	17
3. Translation functions	17
4. Compactness in uniform convergence on compacta	19
5. Compactness in uniform norm	20
6. Bochner version of u.a.p.	22
7. Families indexed by $E^n$	24
8. Composition of AP functions	27
9. Notes	27
Chapter 3: The Fourier Series Theory	29
1. Introduction	29
2. Mean values	30
3. Bessel's inequality	33
4. Mean convergence	36
5. Parseval's equation	38
6. Approximation Theorem	45
7. Differentiation and Integration of Fourier Series	50
8. Differentiability of $a(f, \lambda, x)$	52
9. Notes	54
Chapter 4: Modules and Exponents	56
1. Introduction	56
2. Translation sets	56
3. Kronecker's Theorem	58
4. Module containment	60
5. Convolution by Fourier transforms	65
6. Functions with bounded exponents	67
7. Functions with discrete exponents	68
8. Functions with exponents bounded away from zero	72
9. Notes	75

Chapter 5: Linear Constant Coefficient Equations	77
1. Introduction	77
2. Existence of minimum norm solutions	77
3. The differential equation $y' = f(t)$	78
4. The equation $x' = Ax$	80
5. The $n$ th order scalar equations	81
6. The differential inequalities	83
7. Almost periodic solutions of $x' = Ax$	85
8. The general linear equation	85
9. The scalar equation	87
10. The vector equation again	89
11. Norms of mappings	92
12. Notes	94
Chapter 6: Linear almost periodic equations	97
1. Introduction	97
2. A counterexample	97
3. Bounded solutions	98
4. Favard's Theorem	99
5. Scalar equations	102
6. Linear homogeneous systems	107
7. Periodic homogeneous systems	112
8. Homogeneous systems with bounded fundamental solutions	114
9. Contribution to Question E	115
10. Notes	118
Chapter 7: Exponential Dichotomy and Kinematic similarity	121
1. Introduction	121
2. Exponential dichotomy	121
3. Exponential dichotomy of almost periodic systems	123
4. More on exponential dichotomy	126
5. Row dominant systems	128
6. Column dominant systems	134
7. Kinematic and Approximate similarity	135
8. Notes	140
Chapter 8: Fixed Point Methods	142
1. Introduction	142
2. Applications of Contraction Mapping	142
3. Applications of the Schauder Theorem	144
4. A perturbation result	148
5. Applications to more general equations	151
6. Notes	153

Chapter 9: Asymptotic almost periodic functions and other weaker conditions	154
1. Introduction	154
2. Asymptotic almost periodicity and solutions	154
3. Alternate definitions	155
4. An application	160
5. Extension of almost periodic sequences	163
6. Separation by functionals	166
7. Notes	169
Chapter 10: Separated solutions	170
1. Introduction	170
2. The unique case	170
3. Semi-separated solutions	171
4. A generalization	175
5. Notes	178
Chapter 11: Stable solutions	179
1. Introduction	179
2. Strong stability	179
3. Uniform stability	182
4. Stability of linear systems	190
5. Quasi-stability in the large	196
6. Total stability	199
7. Uniform asymptotic stability	201
8. Periodic equations	205
9. Perturbations of stable systems	209
10. Strong stability again	212
11. Notes	213
Chapter 12: First order equations	217
1. Introduction	217
2. Periodic equations	217
3. Differential equations on a torus	218
4. Ultimate boundedness	223
5. Monotone $f(x,t)$	228
6. Module containment	231
7. Notes	232
Chapter 13: Second order equations	234
1. Introduction	234
2. The maximum principle	234
3. Uniqueness by Lypanov function	237
4. Uniqueness by comparison	246
5. A forced Lienard equation	251
6. Other Lienard equations	258
7. An application of column dominance	259
8. Notes	262

Chapter 14: Averaging	265
1. Introduction	265
2. The averaged equation	265
3. Transformation Lemmas	266
4. The main theorem	274
5. van der Pol example	278
6. Duffing example	279
7. Two timing example	281
8. Notes	284
Chapter 15: The Literature	286
1. Citations	286
2. Bibliography	288
Chapter 16: Index	333
1. Notations	333
2. General index	334