Part II. Towers of fibrations, cosimplicial spaces and homotopy limits

§0. Introduction to Part II

In Part II of these notes we have assembled some results on towers of fibrations, cosimplicial spaces and homotopy limits (inverse and direct) which were needed in our discussion of completions and localizations in Part I, but which seem to be of some interest in themselves. More specifically:

Chapter VIII. Simplicial sets and topological spaces. This chapter does not really contain anything new. It is mainly intended to help make these notes accessible to a reader who knows homotopy theory, but who is not too familiar with the simplicial techniques which we use throughout these notes.

We point out that, in a certain precise sense, there is an equivalence between the homotopy theories of simplicial sets and topological spaces (or CW-complexes); and thus, for homotopy theoretic purposes, it does not really matter whether one uses simplicial sets or topological spaces. To emphasize this, we will throughout these notes (except in Chapter VIII where it might cause confusion) often use the word

space for simplicial set.

Chapter IX. Towers of fibrations. For use in Chapter X, we slightly generalize here two well-known results for a pointed tower of fibrations $\{X_n\}$:

(i) We show that the short exact sequence

$$* \longrightarrow \varprojlim{}^1 \pi_{i+1} X_n \longrightarrow \pi_i \varprojlim X_n \longrightarrow \varprojlim \pi_i X_n \longrightarrow *$$

also exists for $i = 0$. For this, of course, we first have to define a suitable notion of $\varprojlim^1$ for not necessarily abelian groups.

(ii) We generalize the usual homotopy spectral sequence to an "extended" homotopy spectral sequence, which in dimension 1 consists of (possibly non-abelian) groups, and in dimension 0 of pointed sets, acted on by the groups in dimension 1. This we do by carefully analyzing the low-dimensional part of the homotopy sequences of the fibrations $X_n \longrightarrow X_{n-1}$.

At the end of the chapter we show how these results can be used to get information on the homotopy type of the inverse limit space $\varprojlim X_n$.

Chapter X. Cosimplicial spaces. This chapter is concerned with our basic tool: cosimplicial (diagrams of) spaces.

In Part I of these notes (in Chapter I), we defined, for a ring R, the R-completion of a space X as the so-called "total space" of a certain cosimplicial space $\underset{\sim}{R}X$, and in order to prove some of the basic properties of this R-completion we needed, not surprisingly, various results on cosimplicial spaces. Those results are proved in this chapter. We

(i) lay the foundations for a homotopy theory of cosimplicial spaces, and

(ii) combining this with the results of Chapter IX, obtain, for every cosimplicial (pointed) space, an extended homotopy spectral sequence which in many cases (and in particular for $\underset{\sim}{R}X$)

gives useful information on the homotopy type of the total space.

Chapter XI. Homotopy inverse limits. In this chapter we extensively discuss a notion of homotopy inverse limits which gets around the difficulty that, in general, inverse limits do not exist in the homotopy category.

While this is of interest in itself, our main reasons for including a (rather long) chapter on this subject are that:

(i) homotopy inverse limits are closely related to cosimplicial spaces, and the results of this chapter put some of the results of the Chapters IX and X in perspective, and

(ii) we show in this chapter that, up to homotopy, the R-completion of a space X (which was defined in Chapter I as the total space of the cosimplicial $\underset{\sim}{R}X$), is indeed an R-completion of X, in the sense that it is a homotopy inverse limit of the "Artin-Mazur-like" diagram of "target spaces of maps from X to simplicial R-modules"; and this takes (some of) the mystery out of our definition of R-completion.

Moreover we show that:

(iii) the homotopy groups of homotopy inverse limits are quite accessible and there is an extended homotopy spectral sequence for approaching them,

(iv) homotopy inverse limits are closely related to the derived functors $\varprojlim^s$ of the inverse limit functor for abelian groups; and this can be used to extend the definition of $\varprojlim^1$ which we gave in Chapter IX for towers of not necessarily abelian groups, to arbitrary small diagrams,

(v) for a tower of fibrations, the homotopy inverse limit has the same homotopy type as the (ordinary) inverse limit space, and the spectral sequence for the homotopy inverse limit reduces to the short exact sequences of Chapter IX,

(vi) for many cosimplicial spaces (and in particular for $\underset{\sim}{R}X$) the homotopy inverse limit has the same homotopy type as the total space, and the homotopy spectral sequence for the homotopy inverse limit coincides, from E_2 on, with the spectral sequence of Chapter X, and

(vii) there is a cofinality theorem, which enables us to compare homotopy inverse limits for small diagrams of different "shapes", and which we use to show that, for certain large diagrams of spaces, one can, at least up to homotopy, talk of their homotopy inverse limits.

Chapter XII. Homotopy direct limits. Here we briefly discuss the dual notion of homotopy direct limits. We do this mainly for completeness' sake, although a few of the results of this chapter are used in Chapter XI in the proof of (ii).

In writing Part II we have been especially influenced by the work and ideas of Don Anderson and Dan Quillen.