Part I. Completions and localizations

§0. Introduction to Part I

Our main purpose in Part I of these notes, i.e. Chapters I through VII, is to develop for a ring R a functorial notion of R-completion of a space X which

(i) for $R = Z_p$ (the integers modulo a prime p) and X subject to the usual finiteness conditions, coincides, up to homotopy, with the p-profinite completions of [Quillen (PG)] and [Sullivan, Ch.3], and

(ii) for $R \subset Q$ (i.e. R a subring of the rationals), coincides, up to homotopy, with the localizations of [Quillen (RH)], [Sullivan, Ch.2], [Mimura-Nishida-Toda] and others.

Our R-completion is defined for arbitrary spaces, and throughout these notes we have tried to avoid unnecessary finiteness and simple connectivity assumptions. To develop our R-completion we need some homotopy theoretic results on towers of fibrations, cosimplicial spaces, and homotopy limits, which seem to be of interest in themselves and which we have therefore collected in Part II of these notes, i.e. Chapters VIII through XII.

There are, we believe, two main uses for completions and localizations, i.e. for R-completions: first of all, they permit a "fracturing of ordinary homotopy theory into mod-p components"; and secondly, they can be used to construct important new (and old) spaces.

Of course, the general idea of "fracturing" in homotopy theory is very old; and indeed, the habit of working mod-p or using Serre's

$\mathcal{C}$-theory is deeply ingrained in most algebraic topologists. However "fracturing" in its present form (due largely to Sullivan) goes considerably further and, among other things, helps explain the efficacy of the familiar mod-p methods. Roughly speaking (following Sullivan), one can use completions or localizations to "fracture" a homotopy type into "mod-p components" together with coherence information over the rationals; and the original homotopy type can then be recovered by using the coherence information to reassemble the "mod-p components". In practice the rational information often "takes care of itself", and ordinary homotopy theoretic problems (e.g. whether two maps are homotopic or whether a space admits an H-space structure) often reduce to "mod-p problems". Of course, the "world of mod-p homotopy" is interesting in its own right (e.g. see [Adams (S)]).

As remarked above, another use for R-completions is to construct important spaces. It is, in fact, now standard procedure to use localization methods, e.g. Zabrodsky mixing, to construct new finite H-spaces. As other examples, we note that the space $(\Omega^{\infty}S^{\infty})_{(0)}$ is homotopy equivalent to the Z-completion of $K(S_{\infty},1)$, where S_{∞} is the "infinite symmetric group"(see Ch.VII, 3.4), and that, for the Z_p-completion of certain spheres, one can obtain classifying spaces by Z_p-completing suitable non-simply connected spaces (see [Sullivan] and Ch.VII, 3.6). Examples of this sort also seem to be useful in (higher dimensional) algebraic K-theory.

Some more comments are required on the relation between our R-completion and the completions and localizations of others:

In the case $R \subset Q$, as previously noted, our R-completion agrees, up to homotopy, with the <u>localizations</u> proposed by other authors; essentially, we have generalized the localization to non-simply connected spaces.

The situation for $R = Z_p$ is more complicated. Two homotopically equivalent versions of the p-profinite completion have been proposed by [Quillen (PG)] and [Sullivan, Ch.3] for arbitrary spaces; and it can be shown that our Z_p-completion and their p-profinite completion do not coincide, up to homotopy, for arbitrary spaces, although they do for spaces with Z_p-homology of finite type. One difficulty with the p-profinite completion is that for many simply connected spaces (e.g. for $K(M,n)$ where M is an infinite dimensional Z_p-module) the iterated p-profinite completion is not homotopy equivalent to the single one. This difficulty is avoided by the Z_p-completion. Nevertheless, the p-profinite completion remains very interesting, even when it differs from the Z_p-completion.

Some further general advantages of the R-completion are worth mentioning:

(i) Up to homotopy, the R-completion preserves fibrations under very general conditions (namely, when the fundamental group of the base acts "nilpotently" on the R-homology of the fibre).

(ii) Very many spaces X are R-good, i.e. the canonical map from X to its R-completion preserves R-homology and is, up to homotopy, "terminal" among the maps with this property; for instance, if $R \subset Q$ or $R = Z_p$, then all simply connected spaces are R-good, and so are many others (see Chapters V, VI and VII).

(iii) The mod-R homotopy spectral sequence of [Bousfield-Kan (HS)] can be used to relate the R-homology of a space with the homotopy groups of its R-completion.

(iv) The R-completion of a $K(\pi,1)$ has interesting group theoretic significance. For example, the Malcev completion of a nilpotent group π can be obtained as the fundamental group of the Q-completion of $K(\pi,1)$, a fact that suggests how to obtain "Malcev completions with respect to subrings of the rationals" (see Chapter V).

Similarly, the homotopy groups of the Z_p-completions of such a $K(\pi,1)$ have group theoretic significance (see Chapter VI).

Part I of these notes consists of seven chapters, the first four of which deal with the general theory, while the other three are concerned with various applications for $R \subset Q$ and $R = Z_p$. In more detail:

Chapter I. The R-completion of a space. Here we define the R-completion, $R_\infty X$, of a space X, and prove some of its basic properties, such as, for instance, the key property:

(i) A map $X \to Y$ induces an isomorphism on reduced R-homology

$$\tilde{H}_*(X;\ R) \approx \tilde{H}_*(Y;\ R)$$

if and only if it induces a homotopy equivalence between the R-completions

$$R_\infty X \simeq R_\infty Y.$$

Other (not very surprising) properties are:

(ii) The n-type of $R_\infty X$ depends only on the n-type of X.

(iii) Up to homotopy, the R-completion commutes with arbitrary disjoint unions and with finite products, and preserves multiplicative structures.

(iv) There is a generalization to a (functorial) fibre-wise R-completion.

We define $R_\infty X$ by first constructing a cosimplicial diagram of spaces $\underset{\sim}{R}X$, next associating with this a tower of fibrations $\{R_s X\}$, and finally defining the R-completion of X as the inverse limit

$R_\infty X$ of the tower $\{R_s X\}$. Justifications for this definition will be given in Chapters III and XI, where we show that $R_\infty X$ can, in two different ways, be considered as an "Artin-Mazur-like R-completion of X".

A useful tool in handling the R-completion is the homotopy spectral sequence of the tower of fibrations $\{R_s X\}$. This turns out to be the same as the homotopy spectral sequence $\{E_r(X; R)\}$ of X with coefficients in R of Bousfield-Kan, which, for $R = Z_p$, is the unstable Adams spectral sequence, while, for $R = Q$, this spectral sequence consists of the primitive elements in the rational cobar spectral sequence.

At the end of Chapter I we discuss the role of the ring R and show that, for all practical purposes, one can restrict oneself to the rings $R = Z_p$ (p prime) and $R \subset Q$.

Chapter II. Fibre lemmas. For a general fibration of connected spaces $F \to E \to B$, the map $R_\infty E \to R_\infty B$ is always a fibration, but its fibre need not have the same homotopy type as $R_\infty F$. However, there is a mod-R fibre lemma, which states that, up to homotopy, the R-completion preserves fibrations of connected spaces $F \to E \to B$, for which "$\pi_1 B$ acts nilpotently on each $\tilde{H}_i(F; R)$". This condition is, for instance, satisfied if the fibration is principal, or if B is simply connected.

This fibre lemma is a very useful result. It will, for instance, be used in the Chapters V and VI, to compute $\pi_* RX$ in terms of $\pi_* X$, for nilpotent X (i.e. connected X for which, up to homotopy, the Postnikov tower can be refined to a tower of principal fibrations).

Chapter III. Tower lemmas. A convenient feature of our definition of R-completion is its functoriallity. Still, it is often useful to have a more flexible (i.e. up to homotopy) approach available and we therefore prove in this chapter various tower lemmas, which give rather simple sufficient conditions on a tower of fibrations $\{Y_s\}$, in order that it can be used to obtain the homotopy type of the R-completion of a given space X. The strongest of these is the R-nilpotent tower lemma which states roughly:

If $\{Y_s\}$ is a tower of fibrations, together with compatible maps $X \to Y_s$, such that

(i) for every R-module M

$$\varinjlim H^*(Y_s; M) \approx H^*(X; M)$$

(ii) each Y_s is R-nilpotent (i.e. its Postnikov tower can, up to homotopy, be refined to a tower of principal fibrations with simplicial R-modules as fibres),
then, in a certain precise sense, the tower $\{Y_s\}$ has the same homotopy type as the tower $\{R_sX\}$ and hence the inverse limit spaces

$$R_\infty X = \varprojlim R_sX \qquad \text{and} \qquad \varprojlim Y_s$$

have the same homotopy type.

We also observe that $R_\infty X$ is an Artin-Mazur-like R-completion of X, as the results of this chapter imply that, up to homotopy, the tower $\{R_sX\}$ is cofinal in the system of R-nilpotent target spaces of X.

Chapter IV. An R-completion of groups and its relation to the R-completion of spaces. Here we use the greater flexibility of Chapter III, to obtain a more group-theoretic approach to the R-completion. For this we first define an Artin-Mazur-like R-completion of groups, which, for finitely generated groups and $R = Z_p$, reduces to the p-profinite completion of Serre, and which, for nilpotent groups and $R = Q$, coincides with the Malcev completion. Like any functor on groups, this R-completion functor from groups to groups can be "prolonged" to a functor from spaces to spaces, and we show that the latter is homotopically equivalent to the functor R_∞.

As an application we give a very short proof of Curtis' fundamental convergence theorem for the lower central series spectral sequence, at the same time extending it to nilpotent spaces.

Chapter V. Localizations of nilpotent spaces. The main purpose of this chapter is, to show that, for $R \subset Q$, the R-completion of a nilpotent space (i.e. a space for which, up to homotopy, the Postnikov tower can be refined to a tower of principal fibrations) is a localization with respect to a set of primes, and that therefore various well-known results about localizations of simply connected spaces remain valid for nilpotent spaces.

As an illustration we discuss some fracture lemmas (i.e. lemmas which state that, under suitable conditions, a homotopy classification problem can be split into a "rational problem" and "problems involving various primes or sets of primes") and their application to H-spaces.

We also prove that the homotopy spectral sequence $\{E_r(X; R)\}$ converges strongly to $\pi_* R_\infty X$ for $R \subset Q$ and X nilpotent.

Chapter VI. p-completions of nilpotent spaces. This chapter parallels Chapter V: We discuss the p-completion, i.e. the "up to homotopy" version of the Z_p-completion, for nilpotent spaces. This p-completion is merely a generalization of the familiar p-profinite completion for simply connected spaces of finite type, and we prove that various well-known results for such p-profinite completions remain valid for p-completions of nilpotent spaces.

As an illustration we discuss an arithmetic square fracture lemma, which states that, under suitable conditions, a homotopy classification problem can be split into "Z_p-problems" and a "rational problem".

We also obtain convergence results for the homotopy spectral sequence $\{E_r(X; Z_p)\}$ of a nilpotent space X, and observe that the same arguments apply to the lower p-central series spectral sequences.

Chapter VII. A glimpse at the R-completion of non-nilpotent spaces. It is clear from the results of Chapters V and VI that, for nilpotent spaces, the R-completion is quite well understood; however, very little is known about the R-completion of non-nilpotent spaces. In this last chapter of Part I we therefore discuss some examples of non-nilpotent spaces which indicate how much more work remains to be done.

We also make, at the end of this chapter, some comments on possible R-homotopy theories, for $R \subset Q$ and $R = Z_p$.

Warning!!! These notes are written simplicially, i.e. whenever we say

space we mean simplicial set.

However, in order to help make these notes accessible to a reader who knows homotopy theory, but who is not too familiar with simplicial techniques, we will in Chapter VIII, i.e. at the beginning of Part II:

(i) review some of the basic notions of simplicial homotopy theory, and

(ii) try to convince the reader that this simplicial homotopy theory is equivalent to the usual topological homotopy theory. Moreover, we have, throughout these notes, tried to provide the reader with references, whenever we use simplicial results or techniques, which are not an immediate consequence of their well-known topological analogues.

Some of the results of Part I of these notes were announced in [Bousfield-Kan (HR) and (LC)].

In writing Part I we have been especially influenced by the work of Artin-Mazur, Emmanuel Dror, Dan Quillen and Dennis Sullivan.