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The Homology of Iterated Loop Spaces

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Preface

This volume is a collection of five papers (to be referred to as I-V). The first four together give a thorough treatment of homology operations and of their application to the calculation of, and analysis of internal structure in, the homologies of various spaces of interest. The last studies an up to homotopy notion of an algebra over a monad and the role of this notion in the theory of iterated loop spaces. I have established the algebraic preliminaries necessary to the first four papers and the geometric preliminariesnecessary for all of the papers in the following references, which shall be referred to by the specified letters throughout the volume.

[A]. A general algebraic approach to Steenrod operations. Springer Lecture Notes in Mathematics Vol. 168, 1970, 153-231.

[G]. The Geometry of Iterated Loop Spaces. Springer Lecture Notes in Mathematics Vol. 271, 1972.

[G']. E_∞ spaces, group completions, and permutative categories. London Math. Soc. Lecture Note Series Vol. 11, 1974, 61-93.

In addition, the paper II here is a companion piece to my book (contributed to by F. Quinn, N. Ray, and J. Tornehave)

[R]. E_∞ Ring Spaces and E_∞ Ring Spectra.

With these papers, this volume completes the development of a comprehensive theory of the geometry and homology of iterated loop spaces. There are no known results in or applications of this area of topology which do not fit naturally into the framework thus established. However, there are several papers by other authors which seem to me to add significantly to the theory developed in [G]. The relevant references will be incorporated in the list of errata and addenda to [A], [G], and [G'] which concludes this volume.

The geometric theory of [G] was incomplete in two essential respects. First, it worked well only for connected spaces (see [G, p. 156-158]). It was the primary purpose of [G'] to generalize the theory to non-connected spaces. In particular, this allowed it to be applied to the classifying spaces of permutative categories and thus to algebraic K-theory. More profoundly, the ring theory of [R] and II was thereby made possible.

Second, the theory of [G] circumvented analysis of homotopy invariance (see [G, p. 158-160]). It is the purpose of Lada's paper V to generalize the theory of [G] to one based on homotopy invariant structures on topological spaces in the sense of Boardman and Vogt [Springer Lecture Notes in Mathematics, Vol. 347] [1]. In Boardman and

[1] Incidentally, the claim there (p. VII) that [G] failed to apply to non Σ-free operads is based on a misreading; see [G, p. 22].

Vogt's work , an action up to homotopy by an operad (or PROP) on a space was essentially an action by a larger, but equivalent, operad on the same space. In Lada's work, an action up to homotopy is essentially an action by the given operad on a larger, but equivalent, space. In both cases, the expansion makes room for higher homotopies . While these need not be made explicit in the first approach, it seems to me that the second approach is nevertheless technically and conceptually simpler (although still quite complicated in detail) since the expansion construction is much less intricate and since the problem of composing higher homotopies largely evaporates.

We have attempted to make the homological results of this volume accessible to the reader unfamiliar with the geometric theory in the papers cited above. In I, I set up the theory of homology operations on infinite loop spaces. This is based on actions by E_∞ operads $\mathcal{C}$ on spaces and is used to compute $H_*(CX; Z_p)$ and $H_*(QX; Z_p)$ as Hopf algebras over the Dyer-Lashof and Steenrod algebras, where CX and QX are the free $\mathcal{C}$-space and free infinite loop space generated by a space X. The structure of the Dyer-Lashof algebra is also analyzed. In II, I set up the theory of homology operations on E_∞ ring spaces, which are spaces with two suitably interrelated E_∞ space structures. In particular, the mixed Cartan formula and mixed Adem relations are proven and are shown to determine the multiplicative homology operations of the free E_∞ ring space $C(X^+)$ and the free E_∞ ring infinite loop space $Q(X^+)$ generated by an E_∞ space X. In the second half of II, homology operations on E_∞ ring spaces associated to matrix groups are analyzed and an exhaustive study is made of the homology of BSF and of such related classifying spaces as BTop (at $p > 2$) and BCoker J. Perhaps the most interesting feature of these calculations is the precise homological analysis of the infinite loop splitting BSF = BCoker J × BJ at odd primes and of the infinite loop fibration BCoker J → BSF → $BJ_\otimes$ at $p = 2$.

In III, Cohen sets up the theory of homology operations on n-fold loop spaces for $n < \infty$. This is based on actions by the little cubes operad $\mathcal{C}_n$ and is used to compute $H_*(C_nX; Z_p)$ and $H_*(\Omega^n\Sigma^nX; Z_p)$ as Hopf algebras over the Steenrod algebra with three types of homology operations. While the first four sections of III are precisely parallel to sections 1, 2, 4, and 5 of I, the construction of the unstable operations (for odd p) and the proofs of all requisite commutation formulas between them (which occupies the rest of III) is several orders of magnitude more difficult than the analogous work of I (most of which is already contained in [A]). The basic ingredient is a homological analysis of configuration spaces, which should be of independent interest. In IV, Cohen computes

$H_*(SF(n); Z_p)$ as an algebra for p odd and n even, the remaining cases being determined by the stable calculations of II. Again, the calculation is considerably more difficult than in the stable case, the key fact being that $H_*(SF(n); Z_p)$ is commutative even though SF(n) is not homotopy commutative. Due to the lack of internal structure on BSF(n), the calculation of $H_*(BSF(n); Z_p)$ is not yet complete.

In addition to their original material, I and III properly contain all work related to homology operations which antedates 1970, while II contains either complete information on or at least an introduction to most subsequent work in this area, the one major exception being that nothing will be said about BTop and BPL at the prime 2. Up to minor variants, all work since 1970 has been expressed in the language and notations established in I §1-§2 and II §1.

Our thanks to Maija May for preparing the index.

J.P. May
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