Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

491

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Invariants for Real-Generated Uniform Topological and Algebraic Categories

Springer-Verlag
Berlin · Heidelberg · New York 1975

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Library of Congress Cataloging in Publication Data

Broughan, K A 1943-
Invariant for real-generated uniform and algebraic categories.

(Lecture notes in mathematics ; 491)
Bibliography: p.
Includes index.
1. Uniform spaces. 2. Topological spaces.
3. Categories (Mathematics) 4. Invariants.
I. Title. II. Series: Lecture notes in mathematics (Berlin) ; 491.
QA3.L28 no. 491 [QA611.25] 510'.8s [514'.32]
75-34130

AMS Subject Classifications (1970): 02E15, 02E99, 06A10, 10A40, 10B05, 10F35, 10M10, 12J10, 12J20, 13A15, 13J99, 15A03, 18B99, 18D35, 22A99, 26A51, 33A70, 46A15, 54C05, 54C30, 54D20, 54E05, 54E15, 54E25, 54E35, 54F45, 54F50, 54H25.

ISBN 3-540-07418-X Springer-Verlag Berlin · Heidelberg · New York
ISBN 0-387-07418-X Springer-Verlag New York · Heidelberg · Berlin

Offsetdruck: Julius Beltz, Hemsbach/Bergstr.

Introduction

This book is concerned with the construction of natural invariants for categories and algebraic categories of uniform and topological spaces. In each of these categories the natural invariants are similar in definition and form.

The invariants are constructed from the set of values of real valued functions which have generated the structures in each category. An illustration of the method of construction of the invariants follows -- this is the construction in the category of metrizable topological spaces. If a topological space has a compatible metric having its range in a certain subset of the real numbers, then every homeomorphic space has also such a compatible metric. Hence, for each subset of the positive real numbers there will be, at least theoretically, a corresponding metrization theorem, which depends on the structure of the subset chosen. However, not all subsets give rise to unique metrization theorems. If two subsets give rise to the same metrization theorem they are said to be equivalent. More precisely, two subsets of the positive real numbers are called equivalent if each metrizable topological space with a compatible metric having its range in one of the subsets has a compatible metric with range in the other subset, and vice versa. Each equivalence class formed from this relation is a distinct invariant. The set of all equivalence classes thus induced has been given a partial order. One of the objects of this book is to characterize each distinct equivalence class in topological terms. However it is not necessary to deal with the class which is represented by the set of positive real numbers themselves since this characterization is provided by the metrization theorem of Nagata, Smirnov and Bing.

The following is a brief summary of the content of the book.

In Chapter I, the notions of a real-generated category and of an invariant on a category, are defined. The classes of subsets of real numbers are introduced and several theorems are proved about real-generated categories and the related classes. The idea of a derived category is introduced. This notion leads to a method whereby the partially ordered sets of classes of subsets formed from different categories may be related. (This method is used frequently in this book.) Subsets of real numbers are considered to be objects in a category and several morphisms between such objects are constructed. These morphisms are very important for the development of the later theory.

In Chapter II, the special case of uniform spaces is studied. An example of the theorems proved is the following: If a metrizable uniform space has a compatible metric with range in the computable numbers, then the space has a compatible metric with range in the dyadic rationals. It is also proved that the induced partially ordered set of classes of real subsets has an initial segment consisting of a four element chain. Each of the corresponding four equivalence classes is characterized in terms of uniform structures. The class represented by the positive rational numbers is of particular interest. Finally, the theorems for metrizable uniform spaces are extended to arbitrary uniform spaces by using families of pseudometrics.

Invariants for metrizable topological and other types of topological spaces are studied in Chapter III. The following is a typical result: If a space has a compatible metric with range in a closed subset of the real numbers which is not a neighborhood of zero in the positive real numbers, then the space has large inductive dimension zero. This theorem extends earlier results. Then the category of metrizable topological spaces in considered, and the induced partially ordered set of classes of real subsets is shown to have an initial segment consisting of a three element chain. It is not yet known whether the class represented by the positive rational numbers is equal to the third member of this chain. The class of topological spaces having dimension zero is studied in some detail. It is proved that a paracompact space has large inductive dimension zero, if and only if, it has small inductive dimension zero and if every closed-open cover of the space has a closed-open locally finite refinement. Finally, the invariants are studied for classes of topological spaces whose topologies are generated by single real valued functions which satisfy various combinations of axioms, and for spaces whose topologies are generated by families of real valued functions.

In Chapter IV, induced classes are described for topological groups, rings, fields and general topological vector spaces. As the structure of the domain object becomes more complex, the structure generating functions are required to satisfy more stringent requirements -- sufficiently stringent, that is, to ensure that the algebraic operations are continuous. The theorems at each more highly structured level usually depend upon the earlier, more general, results.

In Chapter V, dense subspaces, completions, and the computable generation of spaces are studied. Particular emphasis is given to the classes of spaces introduced in the earlier chapters. Results such as the following are proved: Given any countable subset in any metric space, there exists a metric on the space, uniformly equivalent to the original metric, which has the property that the distance between any two points in the countable subset is rational. It is also shown how any complete separable metric space without isolated points, can be exhibited as the completion of the rational numbers equipped with a rational valued metric compatible with the usual topology. A form of connectedness for uniform spaces is defined and the following theorem proved: A uniform space is connected, if and only if, its completion is connected. A fixed point theorem for functions on the rational numbers is proved and the irrationality of the number "e" is demonstrated.

An index listing the symbols, categories and classes of subsets which are used throughout is included at the end.

My thanks go to Professor Edgar R. Lorch of Columbia University for his advice and encouragement during the preparation of this book.

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