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Automorphic Forms on GL(2)

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Introduction

This is a continuation of "Automorphic Forms on GL(2)". Unfortunately, the reader (if any) will have to have a serious knowledge of the two first chapters of the first volume if he is to find his way through the second one. Perhaps reading Godement's "Notes on Jacquet-Langlands", Institute for Advanced Study (1970) will help him in satisfying this stringent requirement. The main purpose of the second volume is to reformulate and extend a classical result: if

$$\sum a_n/n^s \quad , \quad \sum b_n/n^s$$

are two Dirichlet series associated with automorphic forms (in the classical sense) then the Dirichlet series

$$\sum a_n b_n/n^s$$

is convergent in some right half space, can be analytically continued in the whole complex plane as a meromorphic function of s and satisfies a suitable functional equation. Anything novel in this work comes from the point of view which is the theory of group representations. The local theory in §14 to 18 is a preparation of a technical nature for the global theory of § 19. The motivations appear therefore only in the latter section. The reader should read first §14, take for granted the results of §16 to §18 and then go to §19. §20 is an application to quadratic extensions. Again there is nothing really new in it.

In the Bibliography I have tried to indicate my indebtness to previous authors. But I could not however acknowledge completely my indebtness to G. Shimura. This paper would have never been written if not for a suggestion of his. In particular, the application to quadratic extensions of §20 was, after the oral indications he gave to me, a routine exercise.

I gratefully acknowledge the support of The City University of New York and the National Science Foundation (GP 27952). I wish also to express my thanks to Mrs. Sophie Gerber for typing these notes with competence, patience and understanding.

Finally, I wish to apologize to the mathematical community for presenting a set of notes, if not as bulky, at least as tedious as the first one. My excuse is that I am trying to prove the conjectures outlined by R. Langlands in as many cases as possible. No doubt, the present work will merge into the general case and disappear from the realm of mathematics. I can only hope that somehow, it will sometimes be of some use, however feeble, to the mathematical community.

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Note: This paper is a continuation of "Automorphic Forms on GL(2)", Volume I (Lecture Notes in Mathematics, Volume 114). Unfortunately, the section numbers overlap. Sorry.

Summary and Notations

In general, the notations are the same as in the first volume which is referred to as [1] (see Bibliography). Since the typography is actually different, we give again the principal notations used in [1] as well as a partial list of the new notations introduced in the second volume. We also recall some results which were proved for instance in [8].

The ground field F is a local (commutative) field in Chapter IV (§14 to §18) and an $\underline{A}$-field in Chapter V (§19 and §20). The group G is the group $GL(2)$ regarded as an algebraic group defined over F. We consider the following algebraic subgroups:

$$P = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right\}, \quad N = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right\}, \quad Z = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right\}, \quad A = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\} .$$

Thus Z_F can be identified to $F^\times$. We also set

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} .$$

When F is local we denote by ψ_F or ψ a nontrivial additive character of F and let dx be the self dual Haar measure on F. We denote also by α_F or α the module on F, the module of x being denoted $|x|_F$ or simply $|x|$.

In §14 to §16, the ground field is nonarchimedean. We then denote by q the cardinality of the residual field of F and by v_F or v the normalized valuation. Thus $|x|_F = q^{-v(x)}$. We let R be the ring of integers in F and K be the group $GL(2,K)$.

What we call an Euler factor is a function of s (in $\underline{C}$) of the form $P(q^{-s})^{-1}$ where P is a polynomial such that $P(0) = 1$. In [1] or [8] we associate to each quasi-character χ of $F^\times$ a factor

$\epsilon(s,\chi,\psi_F)$ as well as an Euler factor $L(s,\chi)$. In addition we introduce here the notation

$$\epsilon'(s,\chi,\psi_F) = \epsilon(s,\chi,\psi_F)L(1-s,\chi^{-1})/L(s,\chi) .$$

In [1] we have defined the "irreducible admissible representations" of G_F. If π is such a representation and if it is infinite dimensional it has a Kirillov model noted $\mathcal{K}(\pi,\psi_F)$ here (cf. 2.13 in [1]) as well as a Whittaker model $\mathcal{W}(\pi,\psi_F)$ (cf. 2.14 in [1]). We also introduce an Euler factor $L(s,\pi)$ as well as a factor $\epsilon(s,\pi,\psi_F)$. In addition, we set

$$\epsilon'(s,\pi,\psi_F) = \epsilon(s,\pi,\psi_F)L(1-s,\tilde{\pi})/L(s,\pi) ,$$

where $\tilde{\pi}$ is the representation contragredient to π.

If χ is a quasi-character of F the representation $\pi \otimes \chi$ is defined as being $\pi(g)\chi(\det g)$.

For all integers n we denote by $\mathcal{S}(F^n)$ the space of Schwarz-Bruhat functions on F^n. We also denote by $\mathcal{S}(F^\times)$ the space of locally constant compactly supported functions on $F^\times$. If Φ belongs to $S(F^2)$ we set

$$g.\Phi(x,y) = \Phi[(x,y)g] , \quad z(\chi,\Phi) = \int \Phi(0,t)\chi(t)d^\times t ,$$

where $d^\times t$ is a multiplicative Haar measure and χ a quasi-character. Let π_i, $i = 1,2$ be an irreducible admissible representation of G_F, ω_i the quasi-character of $Z_F = F^\times$ defined by

$$\pi_i(a) = \omega_i(a)1 ,$$

and $\omega = \omega_1\omega_2$. Assuming π_1 and π_2 to be both infinite dimensional, for $\Phi \in \mathcal{S}(F^2)$, W_i in $W(\pi_i,\psi)$, we set

$$\Psi(s,W_1,W_2,\Phi) = \int_{Z_F N_F \backslash G_F} W_1(g)W_2(\eta g)z(\alpha^{2s}\omega, g.\Phi)\,|\det g|^s\, dg\ ,$$

$$\tilde{\Psi}(s,W_1,W_2,\Phi) = \int_{Z_F N_F \backslash G_F} W_1(g)W_2(\eta g)z(\alpha^{2s}\omega^{-1}, g.\Phi)\,|\det g|^s \omega^{-1}(\det g)dg.$$

The analytical continuation of those integrals (which are defined for Res large enough) leads to the definition of an Euler factor $L(s,\pi)$ as well as a factor $\epsilon(s,\pi,\psi)$, where π is the external tensor product $\pi_1 \times \pi_2$. We set

$$\epsilon'(s,\pi,\psi) = \epsilon(s,\pi,\psi)L(1-s,\tilde{\pi})/L(s,\pi)\ .$$

Then we have the functional equation

$$\tilde{\Psi}(1-s,W_1,W_2,\hat{\Phi}) = \omega_2(-1)\epsilon'(s,\pi,\psi)\Psi(s,W_1,W_2,\Phi)\ ,$$

where $\hat{\Phi}$ is defined by

$$\hat{\Phi}(x,y) = \int \Phi(u,v)\psi(yu-xv)du\ dv\ .$$

We conjecture that the factors obey the following rule: (notations are as in [1] §12) if $\pi_i = \pi(\sigma_i)$ where σ_i is a two dimensional representation of the Weil group W_F then

$$L(s,\pi) = L(s,\sigma_1 \otimes \sigma_2)\ ,\quad \epsilon(s,\pi,\psi) = \epsilon(s,\sigma_1 \otimes \sigma_2,\psi)\ ,$$

where the factors in the right-hand side are the ones defined in [9]. Although we fell short of such a goal, all our explicit results are compatible with this assertion (cf. in particular, 19.16).

In §17 the ground field F is $\underset{=}{R}$ the field of real numbers. The group K is the group orthogonal $O(2,\underset{=}{R})$. We have similar notions and results. Of course, we do not consider representations of G_F but rather representations of $\mathcal{H}(G,K)$, the Hecke algebra (cf. [1], §5).

Also we use the following notion of Euler factor. First set

$$G_1(s) = \pi^{-\frac{1}{2}s}\,\Gamma(\tfrac{1}{2}s)\ ,\ G_2(s) = (2\pi)^{1-s}\,\Gamma(s)\ ,$$

where Γ is the gamma function and $\pi = 3.1416.....$. A Euler factor is a function of the form

$$P(s)\prod_i G_1(s+s_i)\prod_j G_2(s+s_j)$$

where P is a polynomial and the s_i , s_j , some constants.

In §18 the ground field F is $\underset{\sim}{C}$ the field of complex numbers. The group K is the group $U(2,\underset{\sim}{C})$. We consider again representations of the Hecke algebra $\mathcal{H}(G,K)$. A Euler factor is now a function of the form

$$P(s)\prod_j G_2(s+s_j)$$

where the s_j are some constants and P is a polynomial.

In Chapter V the ground field F is an $\underset{\sim}{A}$-field. We then follow standard notations and denote by $\underset{\sim}{A}$ the ring of adèles and $\underset{\sim}{I}$ the group of idèles; if v is a place of F then F_v is the corresponding local field and $G_v = GL(2,F_v)$; K_v is the standard maximal compact subgroup of G_v and $K = \prod K_v$. Let ω_i , $i = 1,2$, be two quasi-characters of $\underset{\sim}{I}/F^\times$ and π_i an admissible irreducible representation of the Hecke algebra $\mathcal{H}(G_{\underset{\sim}{A}},K)$ which is contained in the space $G_0(\omega_i)$ (space of cusp forms). Then there is a global Whittaker model $\mathcal{W}(\pi_i,\psi)$ (cf. [1], 9.2). For Φ in $\mathcal{S}(\underset{\sim}{A}^2)$, W_i in $\mathcal{W}(\pi_i,\psi)$ we define as in the local case, two integrals $\Psi(s,W_1,W_2,\Phi)$ and $\tilde{\Psi}(s,W_1,W_2,\Phi)$. They are convergent for $\mathrm{Re}\,s$ large enough, can be analytically continued as a meromorphic function of s and satisfy the functional equation

$$\Psi(s,W_1,W_2,\Phi) = \tilde{\Psi}(1-s,W_1,W_2,\hat{\Phi}) ,$$

where $\hat{\Phi}$ is defined by

$$\hat{\Phi}(x,y) = \int \Phi(u,v)\psi(uy-xv)du\, dv \quad .$$

Here ψ is a nontrivial additive character of $\underline{A}/F$ and du or dv the self dual Haar measure on $\underline{A}$. (Cf. 19.13).

Now we summarize some notations and results used in §19 without further references. Let F_0 and F_1 be two continuous positive functions on $\underline{R}_+^\times$ satisfying the following conditions:

$$F_0 + F_1 = 1 \ , \ F_1(t) = \check{F}_0(t) \quad (= F_0(t^{-1})) \ ,$$

there are t_0 and t_1 such that $0 < t_0 < 1 < t_1$ and

$$F_0(t) = 0 \quad \text{for} \quad 0 < t < t_0 \quad \text{and} \quad F_0(t) = 1 \quad \text{for} \quad t_1 < t \ .$$

If ω is a quasi-character of $\underline{I}/F^\times$ we set

$$\lambda(\omega) = \int_{\underline{I}/F^\times} \omega(a)F_1(|a|)\, da$$

where da is a Haar measure on $\underline{I}/F^\times$. Then if $|\omega| = \alpha_F^s$ with $s > 0$ (α_F denoting the module on $\underline{I}$) the integral is convergent. It is a meromorphic function of ω and $\lambda(\omega) + \lambda(\omega^{-1}) = 0$. The only pole is simple and occurs for $\omega = 1$. If Φ belongs to $\mathcal{S}(\underline{A}^2)$, we define

$$\theta^0(\omega,\Phi) = \int_{\underline{I}/F^\times} \sum_{(\xi,\eta)\neq(0,0)} \Phi(a(\xi,\eta))\omega(a)F_0(|a|)\, da \ ,$$

$$\theta^1(\omega,\Phi) = \int_{\underline{I}/F^\times} \sum_{(\xi,\eta)\neq(0,0)} \Phi(a(\xi,\eta))\omega(a)F_1(|a|)\, da \ ,$$

the sum being extended to all pairs in F^2 except the pair (0,0) .

Then $\theta^0(\omega,\Phi)$ is convergent for all ω and $\theta^1(\omega,\Phi)$ for $|\omega| = \alpha_F^s$ with $s > 2$. Moreover, by Poisson formula,

$$\theta^1(\omega,\Phi) = \theta(\alpha_F^2\omega^{-1},\hat{\Phi}) - \lambda(\alpha_F^2\omega^{-1})\hat{\Phi}(0) - \lambda(\omega)\Phi(0) .$$

Finally the function

$$g \longrightarrow \theta^0(\omega,g.\Phi)$$

where $g.\Phi(x,y) = \Phi((x,y)g)$ is, with the terminology of [1], a slowly increasing function on $G_F \backslash G_{\underline{A}}$. (Cf. [7], VII §5 and [8] §11). We also introduce the subgroup G_0 of g in $G_{\underline{A}}$ such that $|\det g| = 1$ and the subset G'. If F is a number field $G' = G_0$. If F is a function field and the field of constant has cardinality Q we select g_1 such that $|\det g_1| = Q^{-1}$ and set $G' = G_0 \cup G_0 g_1$. Then in both cases $G_{\underline{A}} = Z_{\underline{A}} G'$.

Combining the local and global results for the integrals Ψ and $\tilde{\Psi}$ in the customary fashion, we arrive at the following result which is the main result of the second volume: the representation $\pi = \pi_1 \times \pi_2$ being as above, we set

$$L(s,\pi) = \prod L(s,\pi_v) , \quad \epsilon(s,\pi) = \prod \epsilon(s,\pi_v,\psi_v)$$

where π_v (resp. ψ_v) is the local component of π (resp. ψ, a basic character of $\underline{A}/F$) at the place v; then $L(s,\pi)$ is absolutely convergent in some right half space, can be analytically continued as a meromorphic function of s in the whole complex plane and satisfy the functional equation

$$L(s,\pi) = \epsilon(s,\pi)L(1-s,\tilde{\pi})$$

where $\tilde{\pi}$ is the representation contragredient to π.

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In §20 we let K be a separable quadratic extension of F and denote by $F_{\underline{A}}$ (resp. $K_{\underline{A}}$) the ring of adeles of F (resp. K) and $F_{\underline{A}}^{\times}$ (resp. $K_{\underline{A}}^{\times}$) the group of ideles. If π is now an "irreducible representation" of $GL(2,F_{\underline{A}})$ contained in the space of cusp forms, we associate with π an irreducible representation σ of $GL(2,K_{\underline{A}})$ and show, more or less, that σ is contained in the space of automorphic forms for $GL(2,K_{\underline{A}})$. If w is a place of K above the place v of F, the representation σ_w should be obtained in terms of the representation π_v according to the following rule: assume that $\pi_v = \pi(\tau)$ where τ is a two dimensional representation of the Weil group W_{F_v}, then $\sigma_w = \pi(\tau')$ where τ' is the restriction of τ to the subgroup W_{K_w} of W_{F_v}. Such is the case if v is archimedean. If v is nonarchimedean, we do not know at the moment that all representation of G_v are associated with representation of W_{F_v}, (actually the special representation is not). So we have to introduce an "ad hoc" notion.